FOLDED STRIP LINE OSCILLATOR

E. B. SALSIG

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E B Salsig

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by

Edgar Budd Salsig

Lieutenant, United States Navy

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PREFACE

The material appearing in this thesis was obtained primarily from Stanford University Electronic Research Laboratory Technical Reports. The portions of these technical reports used here were the result of work done in connection with the preparation of Doctors' theses by Ward A. Harmon[6] and Hubert Heffner[3], Research Associates at the ERL. A portion of the mathematical detail was prepared by the writer during his industrial experience tour at the ERL in 1953. Appendix A is original with the writer and was done as an engineering assignment under the guidance and direction of Dr. J. L. Putz, Research Associate at the ERL. The remainder of the work is a graduate level follow-through of the technical report material previously mentioned.

The writer is deeply indebted to Dr. Putz and William Luebke for their patience and zeal in indoctrinating the writer into this complex topic, and especially to Ward A. Harmon, who had previously blazed the trail, analytically speaking, and then painstakingly retraced his steps for the newcomer.

And finally, for her help in preparing the manuscript, I am, as always, indebted to my wife, Doyen.

TABLE OF CONTENTS

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

e = the charge of an electron $= 1.602 \times 10^{-19}$ coulomb

C = gain parameter, $C^3 = \frac{I_0}{8V_0}\frac{E^2}{\beta^2 P}$

$E_{(z)}$ = circuit field

$E_{sc}^{(z)}$ = space charge field

$E_{total}^{(z)}$ = total field acting on an electron

h = plasma wave number $= \frac{e\rho_0}{m u \epsilon}$

H = number of plasma wavelengths in a length L. $H = hL$

$\tilde{i}(z)$ = a.c. conduction current density

i_0 = total d.c. beam current

$K = \frac{1}{2} C^3 \beta \beta_e$

L = the total active length of the circuit

m = the mass of an electron

N = number of wavelengths existing within the tube. $2\pi N = L$

P = complex power

u = total electron velocity

u_0 = d.c. electron velocity

$\tilde{u}(z)$ = a.c. electron velocity

v_p = phase velocity of the cold circuit

V_0 = d.c. beam voltage

β = propagation constant of the cold circuit. The cold circuit waves travel as $e^{j(\omega t - \beta z)}$

$\beta_e = \omega / u_0$

Γ_n = the propagation constant of the interaction waves. In the presence of the beam, the waves travel as Γ_n

ϵ = dielectric constant of free space

LIST OF SYMBOLS (CONTINUED)

η = incremental propagation constant $\Gamma L = -j\beta L + j\lambda$

θ = $(\beta_e - \beta)L$

$\tilde{\rho}(z)$ = a.c. charge density

ρ_0 = d.c. charge density

ω = angular frequency

$\mathbb{E}(s)$ = Laplace transform of E(z)

$I(s)$ = Laplace transform of $\tilde{i}(z)$

LIST OF SYMBOLS (CONTINUED)

[illegible] = incremental propagation constant

[illegible] = ([illegible]$_o$ − [illegible])L

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FOLDED STRIP-LINE OSCILLATOR

1. Introduction

This paper is concerned with the theoretical details of generating microwave energy by using a folded strip-line structure.

The order of treatment will be:

1. Introduction
2. General discussion of periodic structure
3. Wave characteristics used in describing periodic structures
4. Space harmonic components in a folded strip-line periodic structure
5. Theory of backward wave interaction
6. Experimental data and conclusion

Reference material is identified by a superscript numeral. This numeral can be matched to the source listed in the Table of References at the end of the text.

Figure 1-1 is a sketch of a typical folded strip-line structure. A conventional parallel strip-line is also sketched for comparison with the physically folded form. The beam-carrying axial hole is visible. The folded strip-line oscillator is a traveling wave tube and therefore traveling wave tube theory is applicable.

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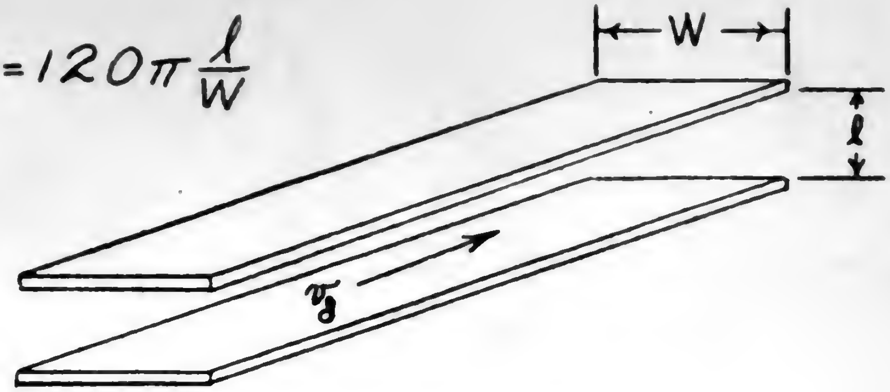
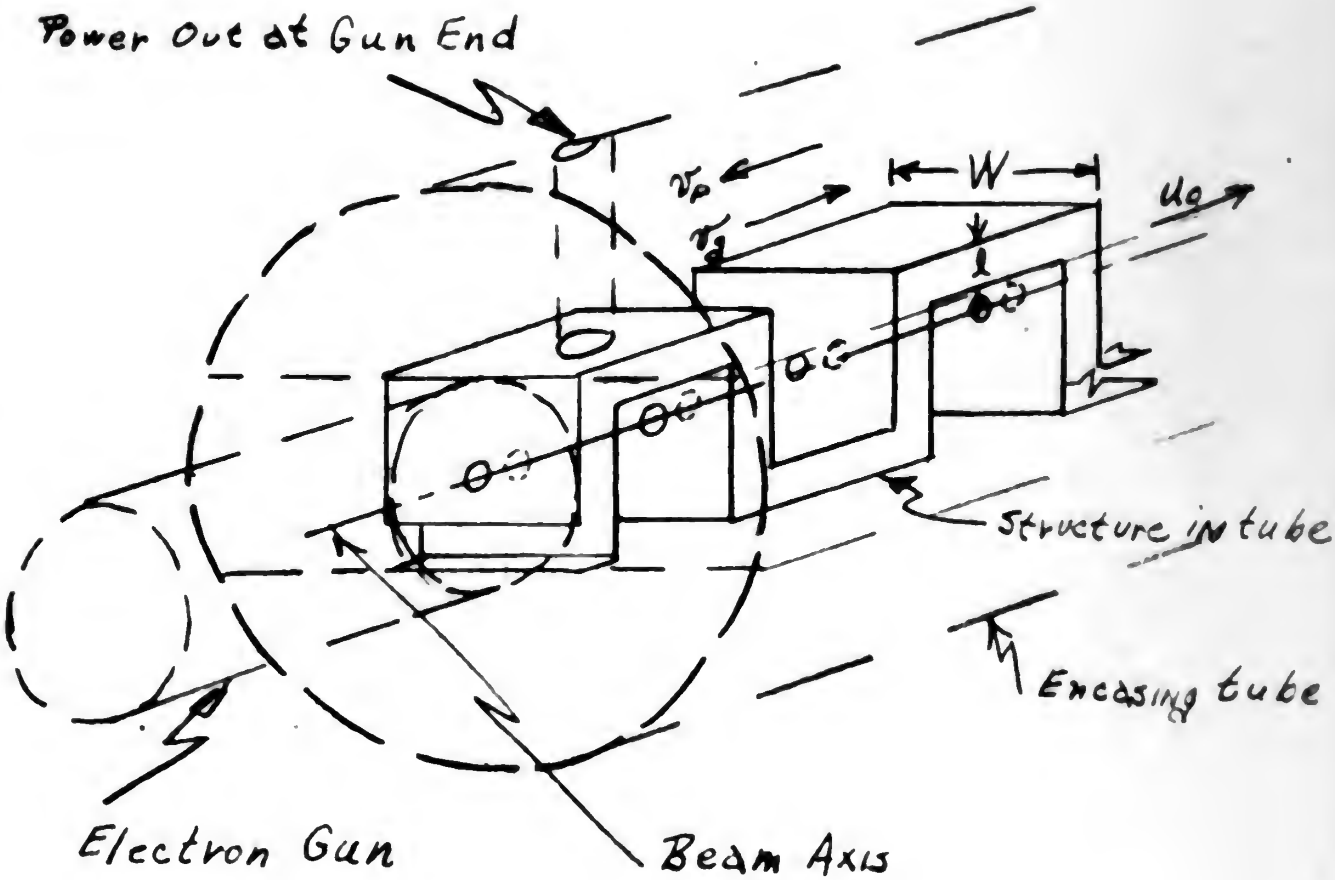
A broad-band electronically controlled oscillator which will be rugged and yet simple to manufacture and operate is needed. The entire art of microwave design has been considerably curtailed by the restricting

$$Z_0 = 120\pi \frac{l}{W}$$



A. Parallel Strip Line



B. Folded Strip Line in tube

FIGURE 1-1 Sketch of Typical Folded Strip Line Tube

characteristics of the three known microwave oscillators: the triode, the magnetron, and the klystron. Until now, generating microwaves above 30 KMC has been regarded as not mechanically feasible. The folded strip-line, backward-wave oscillator shows considerable promise in extending this upper limit far beyond 50 KMC. Further, a single structure design can be made to cover a frequency range in excess of a 3 to 1 frequency ratio, e.g. from 1 KMC to 4 KMC. Expected stability is in the order of 1 part in 10,000, with a CW output in the order of several hundred watts. The only parameter is the electron beam voltage. There is a price, this time in the wide range of beam voltage required, in the high perveance beam necessary, and in the difficulty in focusing the beam.

2. General Discussion of Periodic Structures

Any traveling wave tube oscillator depends upon the energy transfer which occurs from the relatively slow velocity election beam to the slow traveling wave in the periodic structure. The periodic structure is the essential element in the magnetron, the helical traveling wave amplifier, the linear accelerator, and, more recently, in the folded strip-line oscillator.

It is the periodic structure which provides for propagation and yet modifies the manner in which the interacting beam sees the wave. A repetitive, equally-spaced physical change in the structure produces a repetitive change in the propagating wave function. This change in the wave function is periodic, and the generating structure is naturally termed a "periodic structure". As will be seen later, this periodic spatial change will appear as a simple mathematical change in the characteristics of the three known microwave oscillators: the triode, the magnetron, and the klystron. Until now, generating microwaves above 30 KMC has been regarded as not mechanically feasible. The folded strip-line, backward-wave oscillator shows considerable promise in extending this upper limit far beyond 50 KMC. Further, a single structure design can be made to cover a frequency range in excess of a 3 to 1 frequency ratio, e.g. from 1 KMC to 4 KMC. Expected stability is in the order of 1 part in 10,000, with a CW output in the order of several hundred watts. The only parameter is the electron beam voltage. There is a price, this time in the wide range of beam voltage required, in the high perveance beam necessary, and in the difficulty in focusing the beam.

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As a result of the physical modification in the wave form as the beam sees it, the wave may be analyzed into an infinite series of Fourier components which describe the periodic phenomenon[5]. These Fourier components are determined for a given instant of time. The components are for a single frequency. They are due entirely to the physical modification of the wave by the structure. They are termed space harmonics. In order to grasp the concept of a space harmonic it is helpful to reiterate that a single frequency is under consideration, a single time instant is under consideration, and that the equally-spaced, physical disturbances in the periodic propagating structure are entirely responsible for the so-called space harmonics.

From the Fourier analysis it has been shown[5, 6] that there will be an infinite number of space harmonic components for each distinct group velocity. Each space harmonic will have a distinct phase velocity, either forward or backward with reference to the wave group velocity. The space harmonics will have component amplitudes of different magnitudes. The space harmonic components with the greater amplitudes are termed the "principal components", and sometimes "the principal mode"[5].

It is the interaction of the electron beam with the principal components which have a backward phase velocity which results in microwave oscillations.[5, 3] Figure 2-1 is a physical diagram taken from Heffner[3] showing the direction, and the backward phase velocity direction. It can be seen from equation (A-9) in Appendix A that the phase velocity of the principal backward-wave, space-harmonic component is distinct for each exponential definition of the wave function.

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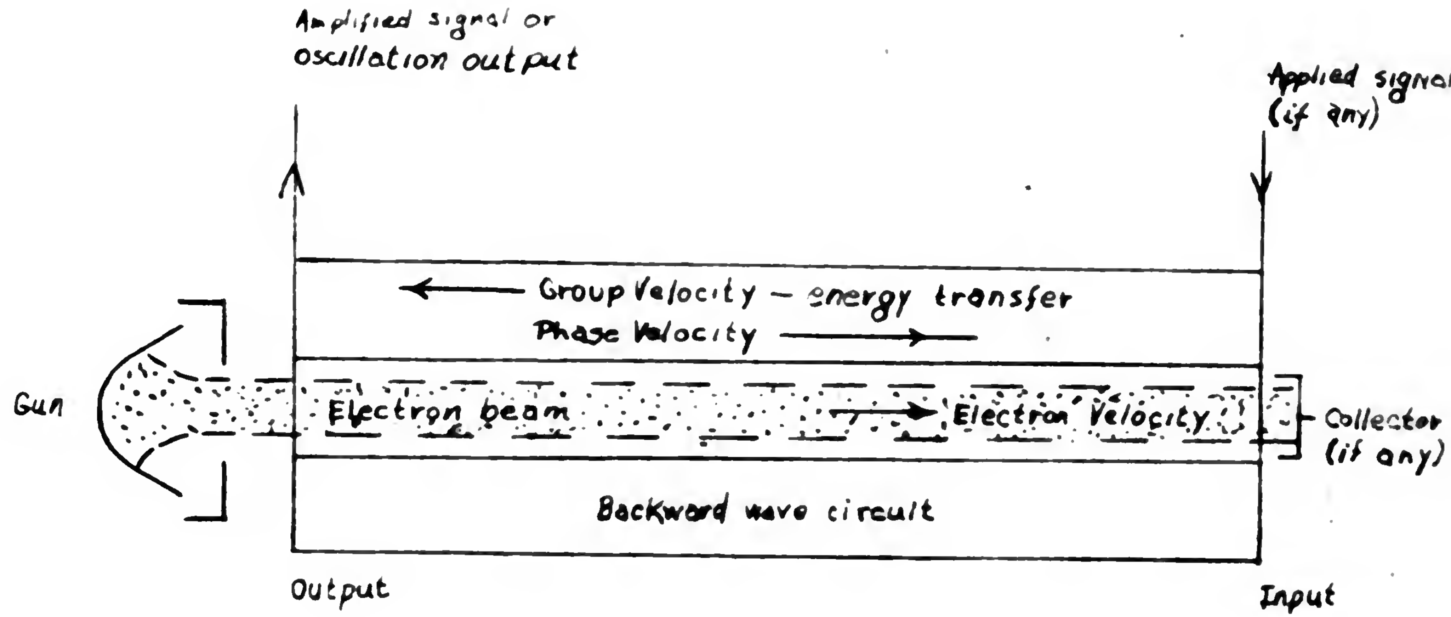

PIGURE 2-1
Schematic diagram of the backward wave tube

distinct frequency. Hence, when the electron beam velocity is matched with the phase velocity of a particular principal space harmonic component, the distinct frequency associated with that principal space-harmonic component will be generated. The beam velocity is directly controlled by the beam voltage, as described in the equation

$$v = 5.93 \times 10^5 \sqrt{V_B} \tag{2-1}$$

where v = meters per second
V_B = potential difference in volts on electrons in the beam

If the beam velocity is subsequently changed, the new beam velocity will interact with a different backward phase velocity. This different phase velocity will be that of a distinct principal space harmonic corresponding to a different frequency. Thus a different frequency will be excited. Change the voltage over a given range, and the frequency will vary over a given range[3]. In this manner, a wide band, tunable microwave oscillator is achieved. It has also been shown that oscillations occur only when beam current is greater than a certain "start oscillation" value[3]. Amplification of the distinct frequency occurs for beam currents less than the "start oscillation" value[5].

3. Wave Characteristics Used in Describing Periodic Structures

A periodic structure is one which will transmit electrical energy but which has physical variations spaced at equal distances down its length[1]. Electrically expressed, these distances are termed periods. The types of periodic structures which may be considered are legion, but the helix is a good example. So is the magnetron structure, or the loaded wave guide.

In fact, the ordinary wave guide with equally-spaced inductive irises is an excellent device for illustrating the development of the periodic structure. It will be used here to illustrate the describing characteristics of the periodic structure. It will also be used to show the relationship which exists between frequency and propagation constant, frequency and phase velocity, frequency and group velocity, and definition of the principal mode[1].

Starting with the ordinary unloaded wave guide, the relationship between the normalized frequency, $\omega\sqrt{\epsilon\mu}$, and the propagation constant β is

$$\omega^2\mu\epsilon = \beta^2 + \beta_c^2 \tag{3-1}$$

where β_c = propagation constant at cutoff frequency

The relationship of (3-1) is shown in Figure 3-1. Note that the ratio of phase velocity to that of light is available from this curve in that

$$\frac{v_p}{v_c} = \frac{\omega\sqrt{\mu\epsilon}}{\beta} \tag{3-2}$$

Further, the slope of the curve gives the ratio of the group velocity to that of light in that

$$\frac{v_g}{v_c} = \frac{\sqrt{\mu\epsilon}\, d\omega}{d\beta} \tag{3-3}$$

For each frequency, β is of the same magnitude but has either a positive or negative sign. The signs and magnitudes of the phase and group velocities merely indicate identical propagation characteristics in either direction.

Now, assume the guide has inductive irises equally spaced at a distance L along its axis. The normalized frequency versus propagation

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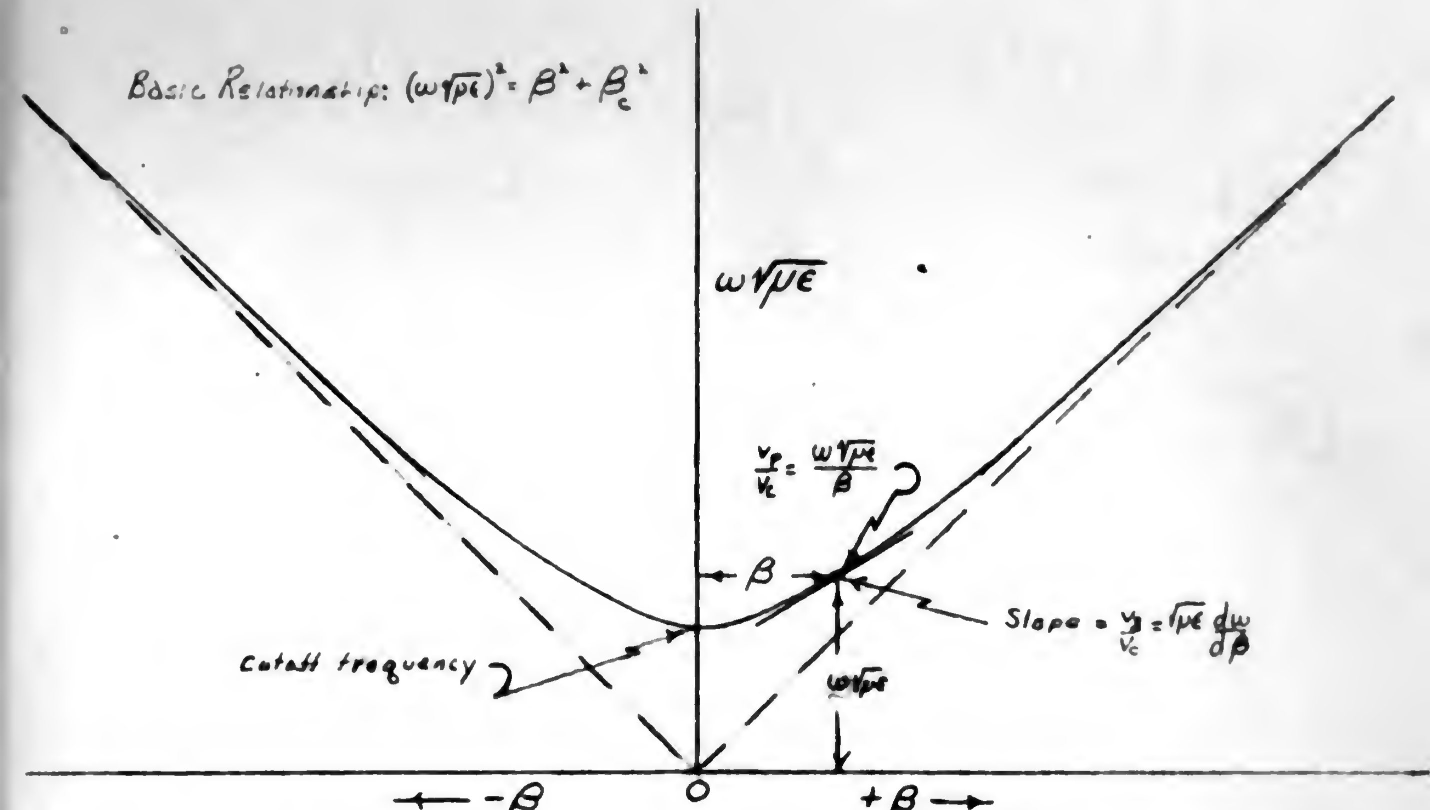

FIGURE 3-1 Propagating Characteristics of an Unloaded Wave Guide

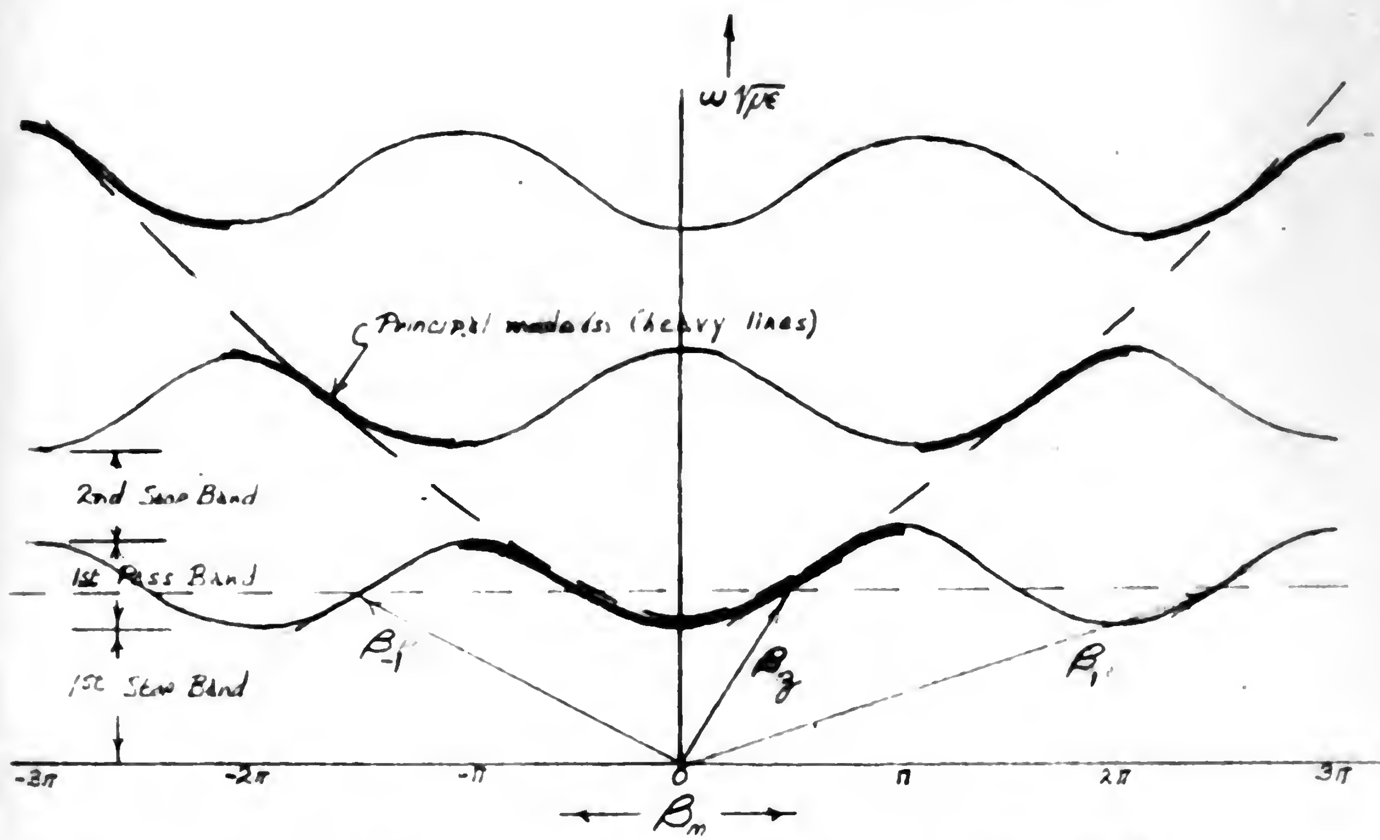

FIGURE 3-2 Propagating Charateristics of Inductively Loaded Wave Guide (Light Loading)

constant curves will appear as in Figure 3-2[1]. The curves indicate that there are pass bands and stop bands in the structure. Further, within a given pass band the curve is definitely repetitive. Since the same relationship still exists for phase velocities and group velocities,

$$\frac{v_p}{v_c} = \frac{\omega\sqrt{\mu\epsilon}}{\beta} \tag{3-2}$$

$$\frac{v_g}{v_c} = \frac{\sqrt{\mu\epsilon}\,d\omega}{d\beta} \tag{3-3}$$

it follows that for a given frequency there are an infinite number of phase velocities and related propagation constants all with the same group velocity. These are the space harmonic components for that given frequency. Furthermore, it follows that there will be an infinite number of phase velocities. This implies waves traveling in the opposite direction to that of the energy propagation. Although it is not apparent from the curves, it is of interest to note that the principal space harmonic components exist in that part of the curve which is most closely related to the unloaded guide[1]. These portions of the curves, which represent the principal components, are drawn extra heavy. The propagation constant, β_n, can be expressed as a periodic function; that is,

$$\beta_n = \beta_z + \frac{2\pi n}{L} \tag{3-5}$$

where L = the periodic spacing in centimeters

$n = \pm$ integer

But it is necessary to define β_z in order to make use of this relationship. Normally, it is chosen as the smallest positive propagation constant, or that of the principal forward space harmonic component[5].

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(3-3) [illegible]

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For the periodically loaded guide it is easy to select β_3 since it corresponds to that β_n which is most nearly equal to β for the unloaded guide. The folded strip-line, which for the non-periodic condition is merely a parallel strip-line, propagates all frequencies at the same velocity. The non-periodic characteristics do not assist in defining β_3. β_3 must be selected from considerations of the principal backward space harmonic, as seen in Appendix A.

To continue in the development of loading an ordinary wave guide, assume the irises to be increased to their limit. Then a series of walled cavities exists down the guide with periodicity L. Figure 3-3 shows this result[1]. For each pass band a single frequency will be propagated. This will be the resonant cavity frequency. This, of course, was to be expected. Multiple resonances are produced which correspond to the limits reached by the pass bands as the iris loading is increased to form a solid wall.

If capacitive rather than inductive irises were used for the loading, similar results might be expected. In fact, in the limiting case, the string of closed cavities produced would be identical. The effect on the pass band curves when capacitive type irises are used as partial loading might occur as in Figure 3-4. This is a speculation by the writer based on observations for the folded strip-line which amounts to capacitive loading. It appears that the general effect of capacitive irises would be to lower the pass band frequencies, while the inductive loading raises the pass band frequencies.

The effect of combined inductive and capacitive loading is not fully understood by the writer, but also appears to be a fertile field for

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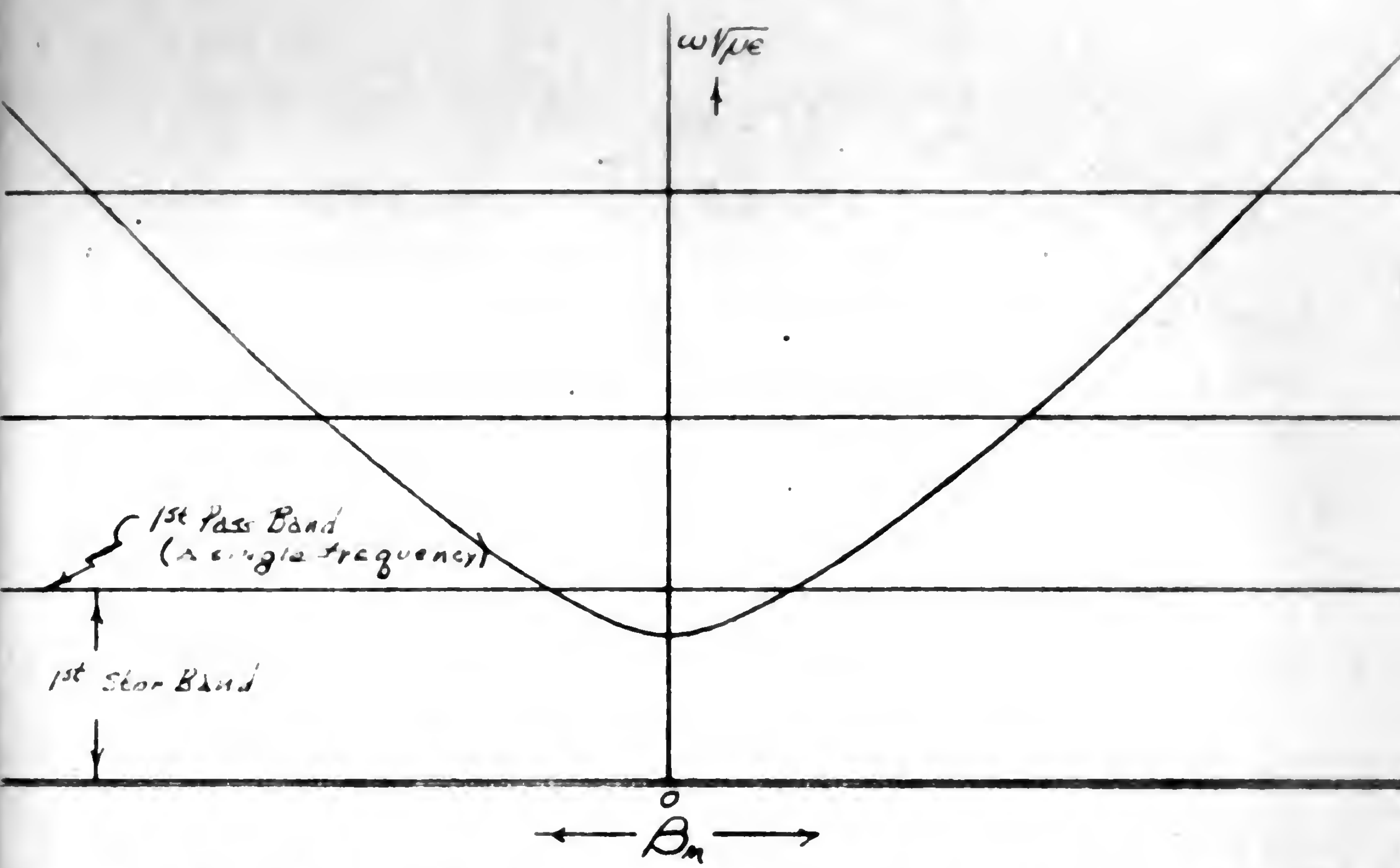

FIGURE 3-3 Propagating Characteristics of a Completely Loaded Wave Guide

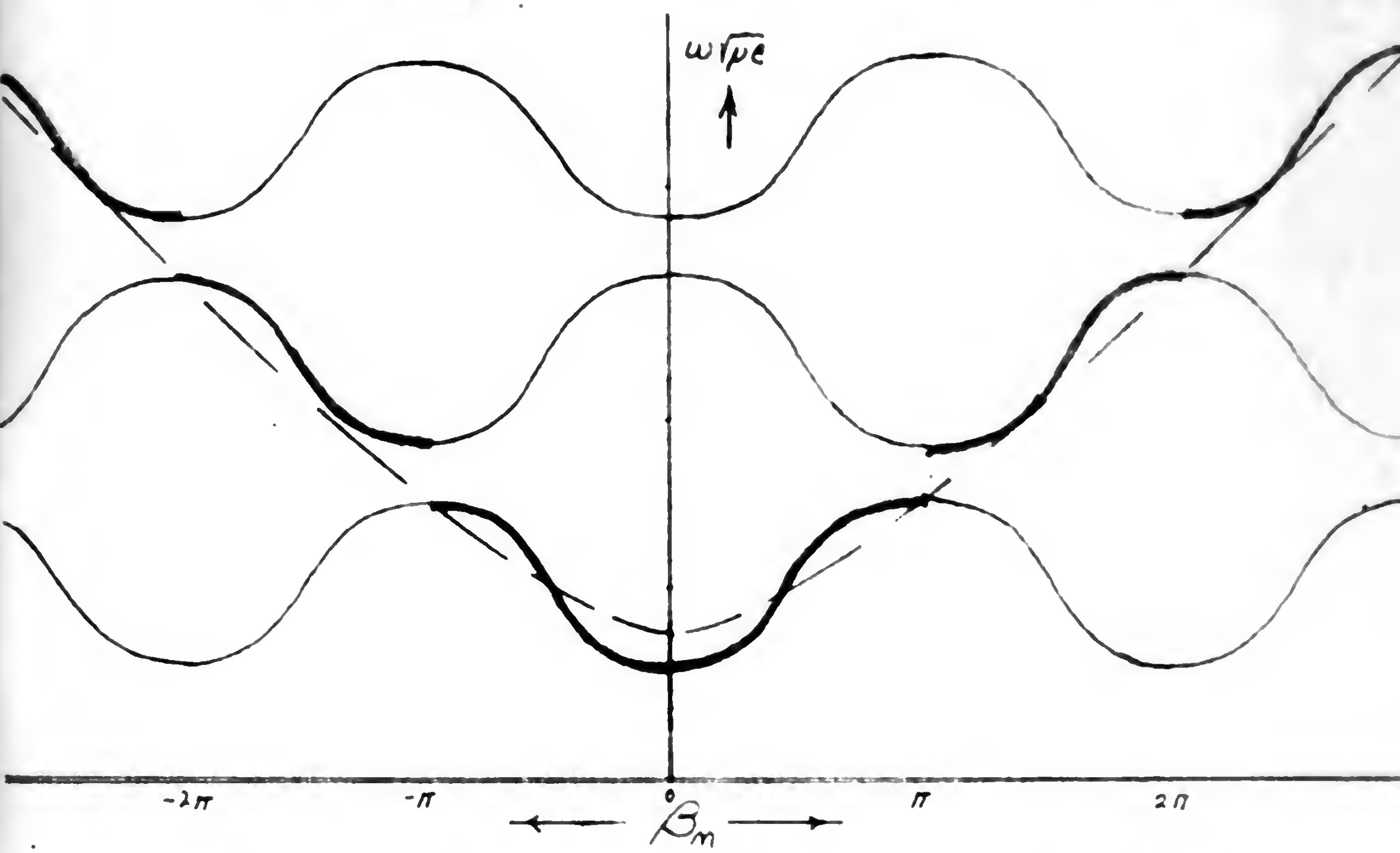

FIGURE 3-4 Propagating Characteristics of Capacitively Loaded Wave Guide (Light Loading)

further investigation. Complete control of the characteristics of periodic structures should be attainable.

The above discussion has presented the concept and a physical example of a periodic structure. It has also introduced the important characteristics, i.e. β_m, β_z, v_p, v_g and $\frac{2\pi m}{L}$ which are used in understanding and mathematically describing the electrical phenomena associated with any periodic structure. The groundwork has been laid for proceeding to the mathematical analyses.

4. Space Harmonic Components in a Folded Strip-line Periodic Structure

It is possible, using Floquet's Theorem and subsequent Fourier analysis, to mathematically describe the space harmonic components which exist at a given frequency in a folded strip-line[1, 5].

Floquet's Theorem states that at a given frequency, for a given mode of oscillation in a periodic structure, the wave function is multiplied by an arbitrary complex constant when traveling down the structure by one period[1]. That is to say, if the wave function of a single space variable z,

$$E(z,t) = E_0 e^{j\omega t - \gamma z} \qquad (4\text{-}1)$$

is propagated through a periodic structure with period L, then Floquet's Theorem states that the wave function in the structure may be expressed as

$$F(z,t) = E_0 \sum_{m=-\infty}^{\infty} e^{j\omega t - \left(\gamma + \frac{2\pi m}{L} j\right) z} \qquad (4\text{-}2)$$

If the structure be lossless, the complex number γ may be written as further investigation. Complete control of the characteristics of periodic structures should be attainable.

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(4-2)

If the structure be lossless, the complex number may be written as

$j\beta_z$, where β_z is a function of frequency alone. The wave function then becomes

$$F(z,t) = E_0 \sum_{n=-\infty}^{\infty} e^{j\left[\omega t - (\beta_z + \frac{2\pi n}{L})z\right]} \qquad (4\text{-}3)$$

If a new phase constant, β_n , is now defined such that

$$\beta_n = \beta_z + \frac{2\pi n}{L} \qquad (3\text{-}5)$$

the wave function may be written

$$F(z,t) = E_0 \sum_{n=-\infty}^{\infty} e^{j(\omega t - \beta_n z)} \qquad (4\text{-}4)$$

It should be noted that the β_n , here defined, is the same β_n discussed previously in the illustrative example of the iris loaded wave guide. The group velocity, v_g , is customarily chosen so that it is positive in the positive Z direction. The wave energy transfer is thereby assumed to be in the positive Z direction. The relationship between v_g and β_n is then

$$v_g = \frac{d\omega}{\partial\beta} \qquad (4\text{-}5)$$

If Floquet's space harmonic series is written as a Fourier series, the magnitude of the components may be evaluated. To solve for the magnitudes, β_z is described in terms of structural parameters. The principal component is also identified. This work is performed in detail in Appendix A.

Since the space harmonic wave function can be expressed as an infinite trigonometric series of period 2π , and since it is continuous over the period with only a finite number of finite discontinuities, it may be written as a Fourier series. Thus,

$$F(z,t) = e^{j\omega t} \sum_{n=-\infty}^{\infty} A_n e^{-j\beta_n z} \tag{4-6}$$

In order to evaluate the coefficients of the space harmonics, the time variation may be regarded as independent of spatial variation. The space harmonic portion of the wave function may then be expressed by

$$f(z) = \sum_{n=-\infty}^{\infty} A_n e^{-j\left(\beta_3 + \frac{2\pi n}{L}\right)z} \tag{4-7}$$

or

$$f(z)\, e^{j\beta_3 z} = \sum_{n=-\infty}^{\infty} A_n e^{-j\frac{2\pi n}{L} z} \tag{4-8}$$

Now, the space harmonic magnitude coefficient, A_n, may be evaluated for each n by the usual methods of Fourier analysis since both sides of equation (4-8) are periodic in 2π.[5] First, multiplying both sides of equation (4-8) by $e^{j\frac{2\pi n}{L}z}$,

$$f(z)\, e^{j\left(\beta_3 + \frac{2\pi n}{L}\right)z} = \sum_{n=-\infty}^{\infty} A_n \tag{4-9}$$

and integrating over a period $L = 2P$,

$$\int_{-P}^{P} f(z)\, e^{j\left(\beta_3 + \frac{2\pi n}{L}\right)z}\, dz = \int_{-P}^{P} A_n\, dz = 2PA_n \tag{4-10}$$

where $n = 0, \pm 1, \pm 2, \ldots \pm \infty$

and solving for A_n,

$$A_n = \frac{1}{2P} \int_{-P}^{P} f(z)\, e^{j\beta_n z}\, dz \tag{4-11}$$

where $n = 0, \pm 1, \pm 2, \ldots \pm \infty$

In Appendix A to this paper, which is original work by the writer, will be found the mathematical details of evaluating the space harmonic

(4-6)

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and integrating over a period in 2 ,

(4-10)

where n

and solving for ,

(4-11)

where n

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$$A_n = \frac{E_0 l}{L} \frac{\sin \frac{\beta_n l}{2}}{\frac{\beta_n l}{2}} \tag{A-11}$$

where n = ± odd integers

and

$$\beta_n = \left[\frac{2\pi v_c}{\lambda_0 v_p}\left(1+\frac{H}{p}\right) + \frac{\pi n}{p} \right] \tag{A-8}$$

where n = ± odd integers

It may be noted from equation (A-11) that the envelope of the possible magnitudes for the space harmonic components in a folded strip-line structure is a $\frac{\sin x}{x}$ type curve. Figure 4-1 shows the shape of such a curve. It is easily seen that the principal space harmonic components with a negative propagation constant will occur somewhere within the range $-\frac{\pi}{2} < \frac{\beta_n l}{2} < 0.$

In fact, it is possible to solve for $\frac{\beta_n l}{2}$ from the relationship governing most efficient interaction between the principal space harmonic component, and the electron beam. Appendix B includes the details of Harmon's[6] work on optimising line spacing, l. The resulting value for $\frac{\beta_n l}{2}$ is equal to 1.166 from which it can be seen, by referring to Figure 4-1, that the optimum value with respect to the mzximized gain parameter, C, also lies within the largest portion of the space harmonic component magnitude envelope. Then l may be expressed in terms of the remaining parameters of the structure as developed in Appendix B, i.e.

magnitude coefficients plus the resulting definition of the space harmonic propagation constant, . Space harmonic component magnitudes are found to be

(A-11)

where n = odd integers

and

(A-12)

where n = odd integers

It may be noted from equation (A-11) that the envelope of the possible magnitudes for the space harmonic components in a folded strip-line structure is a type curve. Figure 4-1 shows the shape of such a curve. It is easily seen that the principal space harmonic components with a negative propagation constant will occur somewhere within the range

In fact, it is possible to solve for from the relationship governing most efficient interaction between the principal space harmonic component, and the electron beam. Appendix B includes the details of Fareman's[6] work on optimizing line spacing, . The resulting value for is equal to 1.165 from which it can be seen, by referring to Figure 4-1, that the optimum value with respect to the maximized gain parameter, C, also lies within the largest portion of the space harmonic component magnitude envelope. Then may be expressed in terms of the remaining parameters of the structure as developed in Appendix B, i.e.,

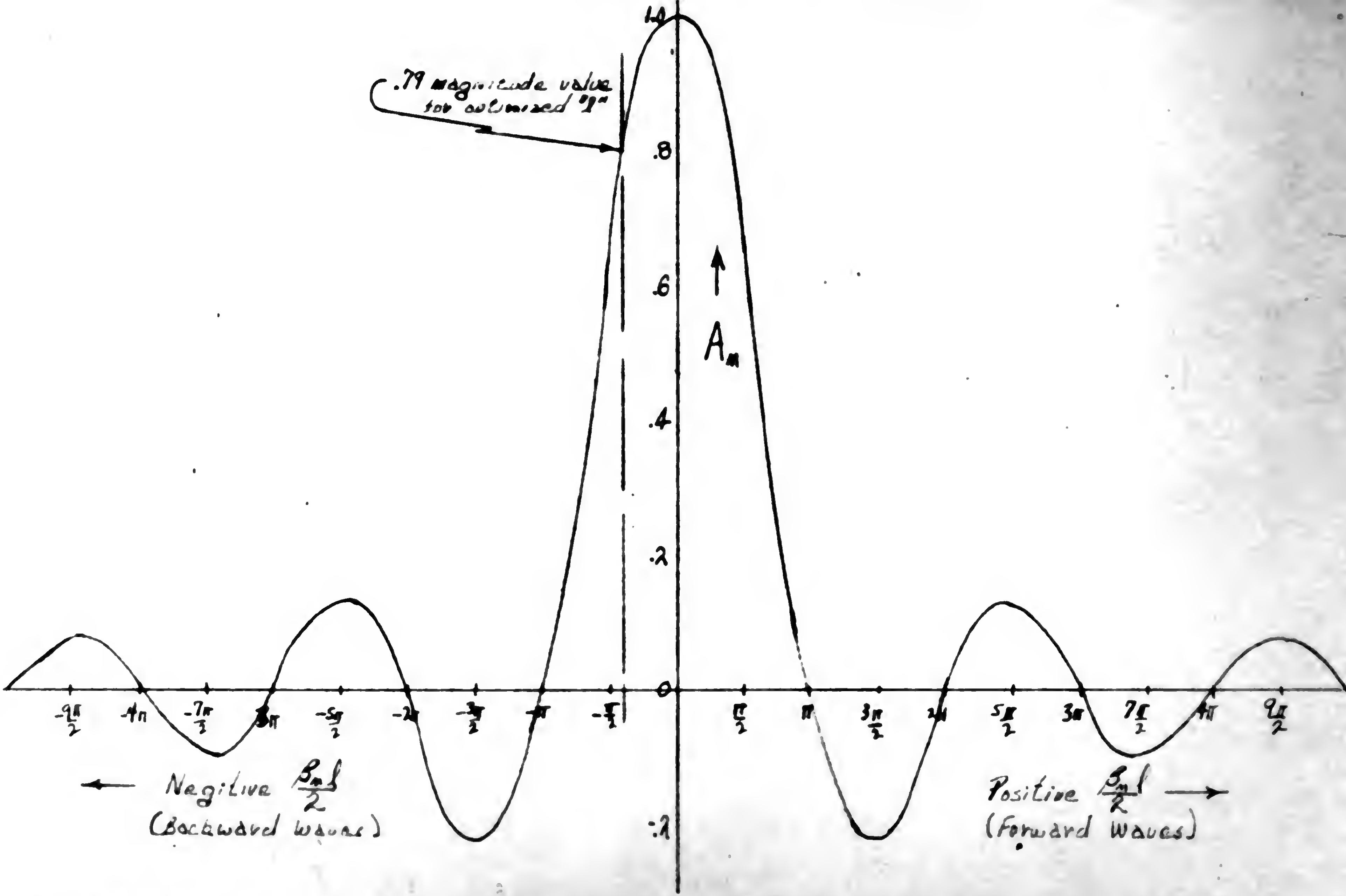

FIGURE 4-1 Normalised Magnitude Envelope of Folded Strip Line Space Harmonic Components

$$l = \frac{.743P}{\frac{2}{\lambda_0}(P+H)+n} \qquad \text{(B-15)}$$

Now, in order to obtain the curves of $\omega\sqrt{\mu\epsilon}$ vs. β_n. for a folded strip-line structure, it is necessary to study the relationship which exists between β_n and λ_0 in equation (A-8) where

$$\beta_n = \frac{2\pi v_c}{\lambda_0 v_p}\left(1+\frac{H}{P}\right)+\frac{\pi n}{P} \qquad \text{(A-8)}$$

where n = ± odd integers

For a given folded strip-line structure, with the structural parameters fixed, equation (A-8) may be expressed,

$$\beta_n P = \frac{D\omega}{v_c}+\pi n \qquad \text{(4-12)}$$

where n = ± odd integers

$$D = \frac{v_c}{v_p}(P+H)$$

Equation (4-12) will be a series of straight lines repeating for different values of n as shown in Figure 4-2 by the solid lines. The dotted lines represent the same series but for negative group velocity[6].

Practically, the structure is enclosed in a tube envelope which produces a low frequency cut-off. Also, the sharp corners at the folds cause stop bands to appear when the value of θ described in equation (A-7):

$$\theta = \frac{2\pi v_c}{\lambda_0 v_p}(P+H) \qquad \text{(A-7)}$$

goes through values equal to an integral number of half wave lengths of the frequency considered. Figure 4-3 presents the actual $\frac{D\omega}{v_c}+\pi n$ vs. β_n plot for a folded strip-line tube for positive group velocities[7].

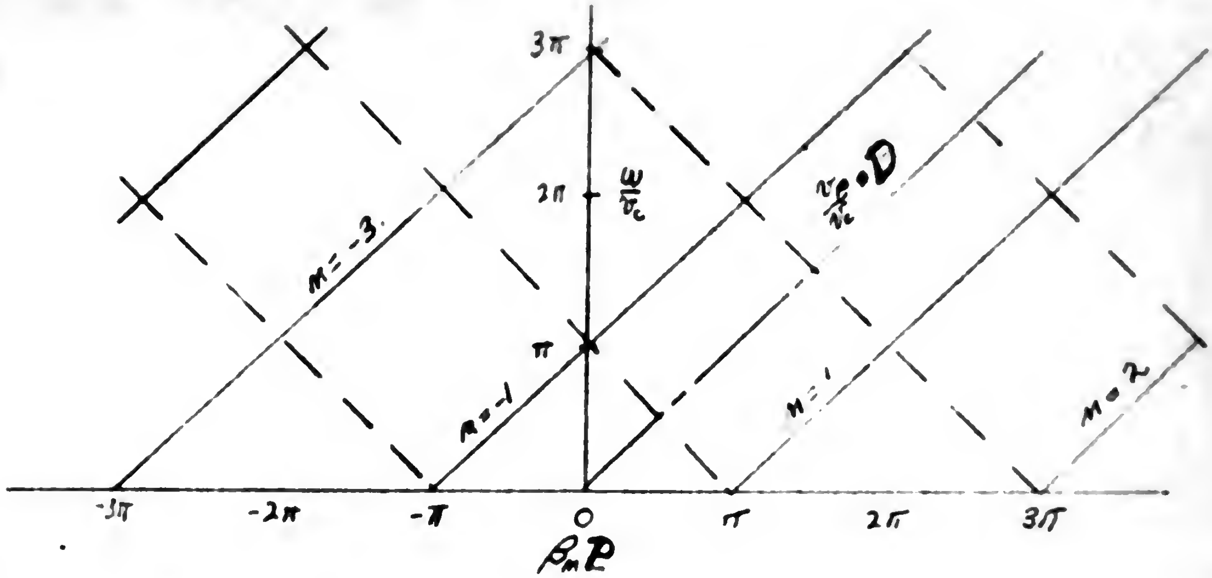

FIGURE 4-2 Theoritical Propagating Characteristics for Folded Strip Line Structure

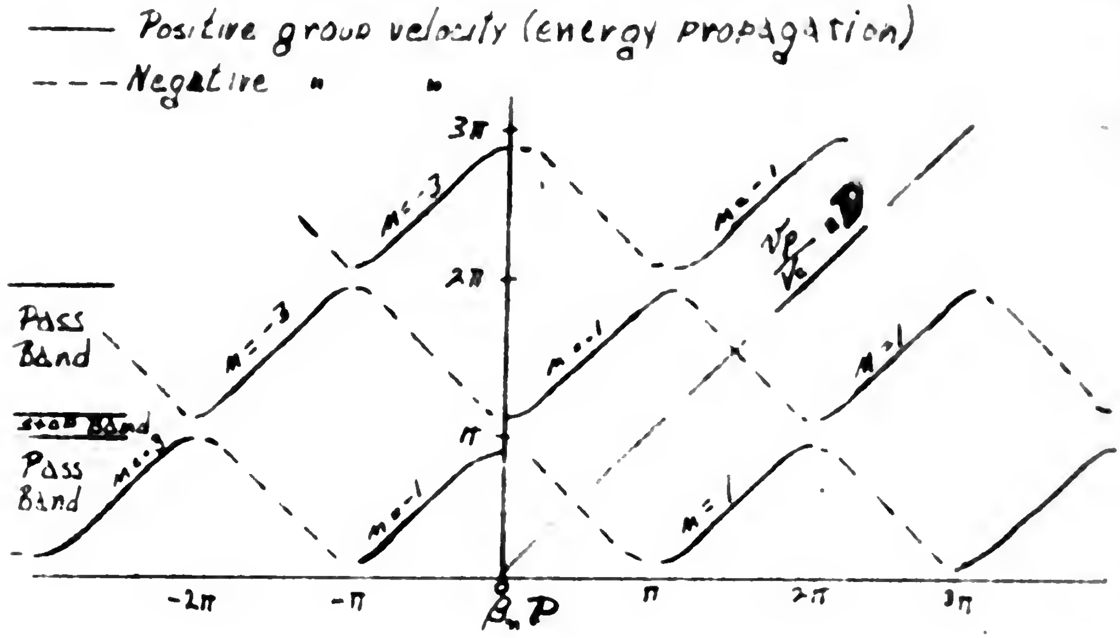

FIGURE 4-3 Actual Propagating Characteristics for Folded Strip Line Structure

Again, the phase velocity is proportional to the slope of the actual curves. At a given frequency, it will be noted that the group velocity for all the space harmonic components, that is all β_n's, is the same, but that phase velocity is either positive or negative depending on mode number.

Some further relationships regarding performance characteristics in terms of structure parameters are developed in Appendix B. The first is the relationship between beam voltage, V_B , and principal component propagation constant, β_n , which was shown as

$$\beta_n = \frac{2\pi \times 506}{\lambda_0 \sqrt{V_B}} \tag{B-19}$$

The gain parameter was also expressed in terms of structure parameters, and was shown to be[2, 6],

$$C_n = .0266 \left[\frac{M \lambda_0}{L} \right]^{2/3} \left[\frac{I_B l}{W} \right]^{\frac{1}{3}} \tag{B-23}$$

where $M = \dfrac{\sin \frac{\beta_n l}{2}}{\frac{\beta_n l}{2}}$

Thus, mathematically, the nature of the space harmonic components in a folded strip-line structure has been determined, and the relationship between these components and structural parameters has been developed so that a reasonable insight may be gained into tube design and operation.

5. <u>The Theory of Backward Wave Interaction</u>

Mathematical analysis of backward wave interaction is defined in terms of small signal theory[3, 6]. The current induced in the beam is related to the structure field. This is called the Electronics Equation. Then the field induced on the structure is related to the current in beam.

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(B-19)

The gain parameter was also expressed in terms of structure parameters, and was shown to be[2, 6],

(B-23)

where H

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This is called the Circuit Equation. The Electronic and Circuit Equations are then combined into a single fourth order differential equation. The solution to the combined equation will be in terms of four separate fields[3, 6]. It has been shown that one of these four defining fields is insignificant in magnitude[3, 6].

The details of the Electronic and Circuit Equations, their combination, and their solution are contained in Appendix C to this paper. Appendix C is a mathematically complete follow-through of Heffner's and Harmon's work in this field[3, 6]. The one exception is the solution to the characteristic equation (C-60) and subsequent derivation of (C-74). This the writer could not verify in the time available. Therefore, the results obtained by Heffner[3] are indicated.

The solution to the combined equation when only three of the four describing waves are considered is

$$E_{(z)} = \frac{\dfrac{\delta_3-\delta_2}{\delta_1}e^{j\delta_1\frac{z}{L}} + \dfrac{\delta_1-\delta_3}{\delta_2}e^{j\delta_2\frac{z}{L}} + \dfrac{\delta_2-\delta_1}{\delta_3}e^{j\delta_3\frac{z}{L}}}{\dfrac{\delta_3-\delta_2}{\delta_1}e^{j\delta_1} + \dfrac{\delta_1-\delta_3}{\delta_2}e^{j\delta_2} + \dfrac{\delta_2-\delta_1}{\delta_3}e^{j\delta_3}} \qquad \text{(C-74)}$$

where the roots $\delta_{1,2,3}$ are determined from the characteristic equation

$$\delta^3 + 2\theta\delta^2 + (\theta^2 - H^2)\delta + \frac{K}{2\beta} = 0 \qquad \text{(C-73)}$$

Equation (C-74) actually is the expression for backward wave gain. It can be seen that gain is an interference phenomenon[3, 6]. The three waves composing the field must produce a net field which is equal to the applied field at the input end. It can be stated that because of this interference phenomenon, one or more of the three component fields may be

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[illegible] (C-74)

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[illegible] (C-75)

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The start oscillation conditions are defined by setting the denominater of equation (C-74) equal to zero, and solving for the roots. This yields the two equations

$$\cos\left(\theta+\frac{3\eta_1}{2}\right)+\frac{2\eta_1(\theta+\eta_1)}{(\theta+\eta_1)^2-H^2}\cosh\sqrt{\theta\eta_1+\frac{3\eta_1^2}{4}-H^2}=0 \qquad \text{(C-77)}$$

$$\sin\left(\theta+\frac{3\eta_1}{2}\right)+\frac{\theta(\theta+\eta_1)+H^2}{(\theta+\eta_1)^2-H^2}\left[\frac{\eta_1}{\sqrt{\theta\eta_1+\frac{3}{4}\eta_1^2-H^2}}\right]\sinh\sqrt{\theta\eta_1+\frac{3\eta_1^2}{4}-H^2}=0 \qquad \text{(C-78)}$$

Heffner[3] described the manner in which these equations may be solved in terms of line parameters and showed that the start oscillation conditions take the form which requires that

$$(\beta-\beta_e)L = \text{a constant}$$

and that

$$CN = \text{a constant}$$

This means that there exists a frequency and current which first allows finite fields to exist on the structure in the absence of any applied field. As the beam velocity is changed, the frequency must also change in order that the synchronous velocity difference, $(\beta-\beta_e)L$ remain constant.

larger than the applied field. Heffner[3] has shown that for the start oscillation condition, with the field reduced to zero, finite fields still exist on the structure. At the input end the three describing fields now add up to a zero net field. Figure 5-1 illustrates Heffner's interpretation of backward wave oscillation where zero net field exists at the input (collector) end, while a finite net field results at the output (gun) end[3].

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(6-77)

(6-78)

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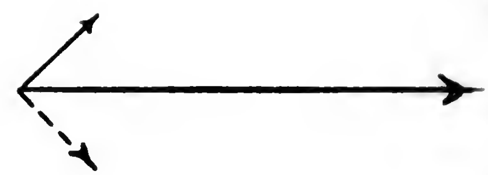
= a constant

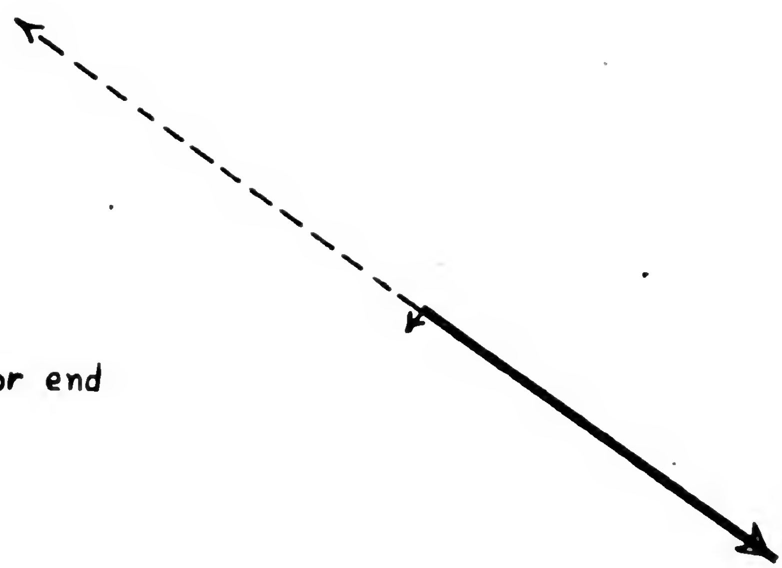
and that

= a constant

This means that there exists a frequency and current which first allows finite fields to exist on the structure in the absence of any applied field. As the beam velocity is changed, the frequency must also change in order that the synchronous velocity difference, remain constant.

a) Gun end





b) Collector end

FIGURE 5-1 Relative Magnitudes of the Three Defining Waves for the Start Oscillation Condition

Since the beam velocity is proportional to the square root of the beam voltage, electronic tuning results. Heffner has shown[3] that for oscillations to exist, the beam current must be greater than a computable "start oscillation" value. He has also shown that for predictable electronic tuning the beam current must remain constant.

6. Experimental Data and Conclusion

It has been shown how a slow-traveling backward wave is produced using periodic disturbances in a propagating system. The nature of the space harmonics on the folded strip-line has been mathematically analyzed. Finally, the interaction of the structure field and electron beam has been analyzed in order to show that oscillation is an interference phenomenon. Oscillations could be regarded as the result of incremental feedback with built in phase compensation. The dependency of oscillation frequency on beam voltage has also been shown. At the Stanford University Electronics Research Laboratory, Putz and Luebke have constructed several folded strip-line structures and tested them both as amplifiers and more lately as oscillators, and have verified the theory discussed herein[5, 7]. Although military security restrictions prevent the use of exact quantities, relative curves of performance are presented in Figure 6-1. These curves, showing relative frequency vs. relative beam voltage for several relative beam currents, may be observed to display the characteristics theorized above. These curves represent operation with the principal backward mode being that where $n = -1$, and for the first pass band.

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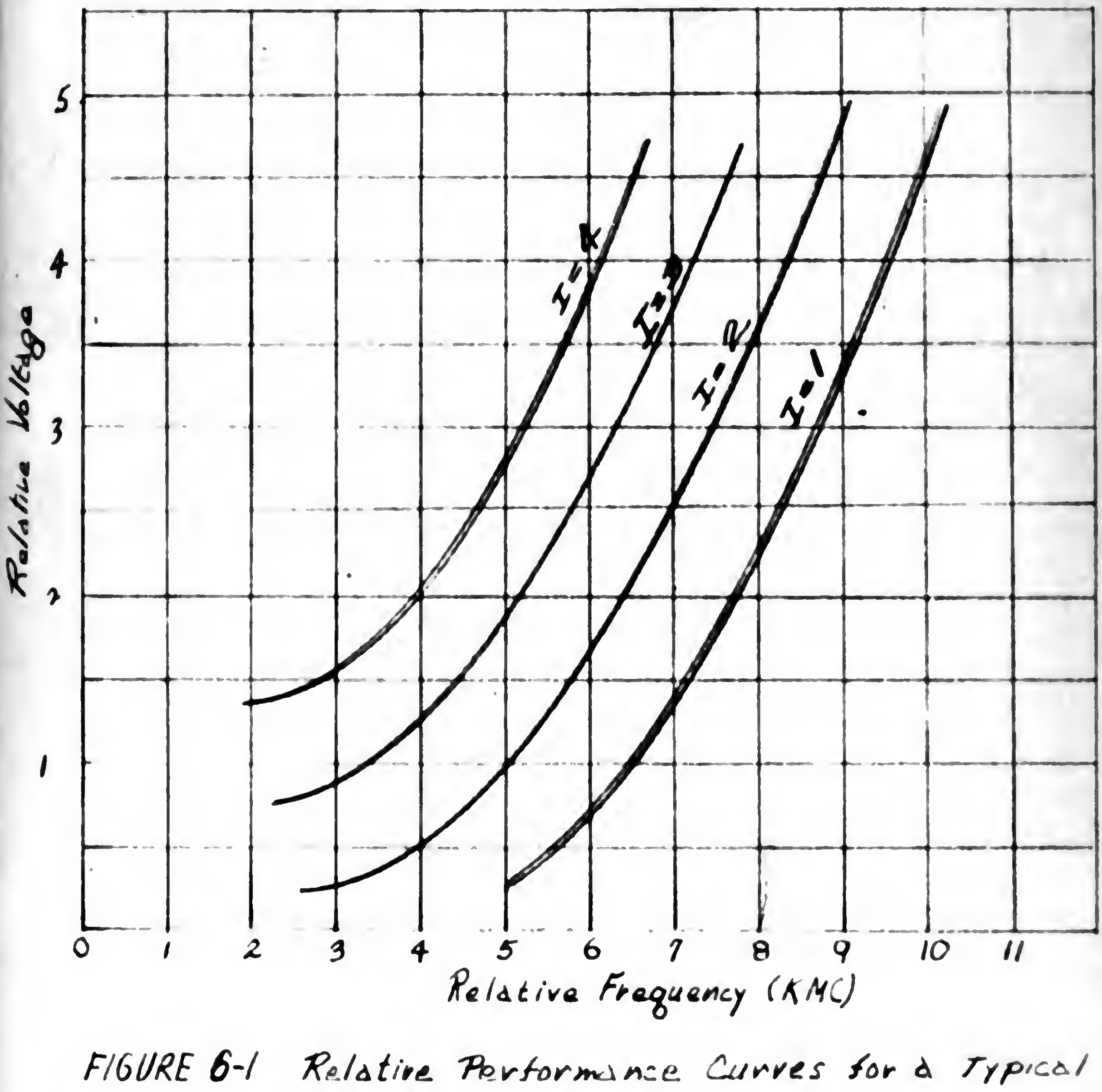

FIGURE 6-1 Relative Performance Curves for a Typical Folded Strip Line Oscillator

In conclusion, it may be predicted that as development on this and other space harmonic oscillators continues, microwave techniques may soon feel the impetus of a new development no less significant than that of the transistor in connection with low frequency circuitry.

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APPENDIX A

Evaluation of the Fourier Coefficients for the Space Harmonic Components in a Folded Strip-Line Structure

The evaluation of the Fourier coefficient which expresses wave propagation in a folded strip-line structure, Equation (4-11)

$$A_n = \frac{1}{2P}\int_{-P}^{P} f(z)\, e^{j\beta_n z}\, dz \qquad (4\text{-}11)$$

requires that $f(z)$ be evaluated within the period -P to P. Figure A-1 is a schematic of the folded strip-line with appropriate dimension and with the axial coordinate directions.

Within the period -P to P, $f(z)$ will then be:

(a) From $-P$ to $-(P-\frac{\ell}{2})$, $\quad f(z) = -E_0 e^{j\theta}$

(b) From $-\frac{\ell}{2}$ to $\frac{\ell}{2}$, $\quad f(z) = E_0$

(c) From $P-\frac{\ell}{2}$ to P, $\quad f(z) = -E_0 e^{-j\theta}$

where θ = phase shift between successive gaps.

Then substituting in Equation (4-11),

$$A_n = \frac{E_0}{2P}\left\{-\int_{-P}^{-(P-\frac{\ell}{2})} e^{j\theta} e^{j\beta_n z}\, dz + \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} e^{j\beta_n z}\, dz - \int_{P-\frac{\ell}{2}}^{P} e^{-j\theta} e^{j\beta_n z}\, dz\right\}$$

$$= \frac{E_0}{j2\beta_n P}\left\{-e^{j\theta} e^{j\beta_n z}\Big|_{-P}^{-(P-\frac{\ell}{2})} + e^{j\beta_n z}\Big|_{-\frac{\ell}{2}}^{\frac{\ell}{2}} - e^{-j\theta} e^{j\beta_n z}\Big|_{P-\frac{\ell}{2}}^{P}\right\}$$

$$= \frac{E_0}{j2\beta_n P}\left\{-e^{j\theta} e^{-j\beta_n(P-\frac{\ell}{2})} + e^{j\theta} e^{-j\beta_n P} + e^{j\beta_n\frac{\ell}{2}} - e^{-j\frac{\beta_n \ell}{2}} - e^{j\theta} e^{j\beta_n P} + e^{-j\theta} e^{j\beta_n(P-\frac{\ell}{2})}\right\}$$

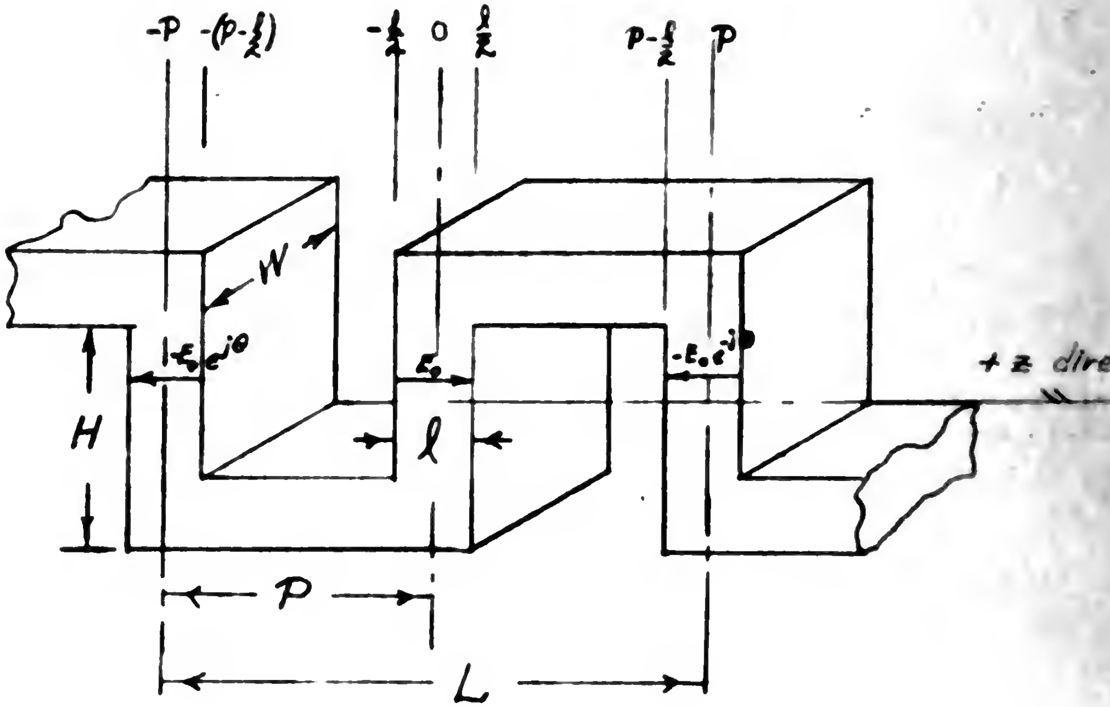

FIGURE A-1 Schematic of Folded Strip Line

$$A_n = \frac{E_0}{\beta_n P}\left\{\frac{e^{j\beta_n \frac{l}{2}} - e^{-j\beta_n \frac{l}{2}}}{2j} - \frac{e^{j(\beta_n P-\theta)} - e^{-j(\beta_n P-\theta)}}{2j} + \frac{e^{j[\beta_n(P-\frac{l}{2})-\theta]} - e^{-j[\beta_n(P-\frac{l}{2})-\theta]}}{2j}\right\}$$

$$= \frac{E_0}{\beta_n P}\left\{\sin\beta_n l - \sin(\beta_n P-\theta) + \sin[\beta_n(P-\tfrac{l}{2})-\theta]\right\}$$

$$= \frac{E_0}{\beta_n P}\left\{\sin\frac{\beta_n l}{2} - \sin(\beta_n P-\theta) + \sin\left[\frac{\beta_n l}{2} - (\beta_n P-\theta)\right]\right\} \quad (A-1)$$

Now, in order to further simplify and subsequently solve for A_n, let

$$\sin(\beta_n P-\theta) = 0 \qquad (A-2)$$

This requires that

$$\beta_n P - \theta = \pi n \qquad (A-3)$$

where $n = 0, \pm 1, \pm 2, \ldots \pm \infty$

and then

$$\beta_n = \frac{\theta}{P} + \frac{\pi n}{P} \qquad (A-4)$$

but, by definition

$$\beta_n = \beta_z + \frac{2\pi n}{L} = \beta_z + \frac{\pi n}{P} \qquad (3-4)$$

(Selected period L = 2P in this case)

This requires that

$$\beta_z = \frac{\theta}{P} \tag{A-5}$$

It is desirable to express β_z in terms of the parameters of the structure. This may be done if it is assumed that the wave energy propagates within the strips. This will occur if the strips are wide enough and the frequency high enough. From experience this condition exists for parallel strip lines where the characteristic impedance is less than 100 ohms. The characteristic impedance is expressed as

$$Z_o = 120\pi \frac{\ell}{w} \tag{A-6}$$

where ℓ = line spacing
and w = line width

It is also assumed that there are no other conductors in the vicinity of the fringing fields, such as placing a folded strip in a wave guide. This is exactly what is done when building the oscillator. It does not appreciably change the results obtained, however, if the sides of the enclosing wave guide are reasonably remote from the structure. It is further assumed that there are no effects due to the sharp corners at the bends.

The phase shift between gaps, θ, is regarded entirely as if the wave were propagating down an unfolded strip-line. It may be expressed

$$\theta = \frac{2\pi v_c}{\lambda_o v_p}(P+H) \tag{A-7}$$

where v_c, v_p, λ_o are unfolded strip-line wave propagation characteristics at the given frequency

and P, H are folded line structure parameters (see Fig. A-1)

This requires that

(A-5)

It is desirable to express in terms of the parameters of the structure. This may be done if it is assumed that the wave energy propagates within the strips. This will occur if the strips are wide enough and the frequency high enough. From experience this condition exists for parallel strip lines where the characteristic impedance is less than 100 ohms. The characteristic impedance is expressed as

(A-6)

where line spacing
and line width

It is also assumed that there are no other conductors in the vicinity of the fringing fields, such as placing a folded strip in a wave guide. This is exactly what is done when building the oscillator. It does not appreciably change the results obtained, however, if the sides of the enclosing wave guide are reasonably remote from the structure. It is further assumed that there are no effects due to the sharp corners at the bends.

The phase shift between gaps, , is regarded entirely as if the wave were propagating down an unfolded strip-line. It may be expressed

(A-7)

where are unfolded strip-line wave propagation characteristics at the given frequency

and , N are folded line structure parameters (see Fig. A-1)

Then,

$$\beta_3 = \frac{2\pi v_c}{\lambda_o v_p}\left[1 + \frac{H}{P}\right] \tag{A-8}$$

where $\left[1 + \frac{H}{P}\right]$ is defined as the slowdown factor of the structure[5]

Then β_n becomes,

$$\beta_n = \frac{2\pi v_c}{\lambda_o v_p}\left[1 + \frac{H}{P}\right] + \frac{\pi n}{P} \tag{A-9}$$

Now, returning to equation (A-1) and proceeding with its solution,

$$A_n = \frac{E_o}{P\beta_n}\left\{\sin\frac{\beta_n l}{2} - \sin(\beta_n P - \theta) - \sin\frac{\beta_n l}{2}\cos(\beta_n P - \theta) + \cos\frac{\beta_n l}{2}\sin(\beta_n P - \theta)\right\} \tag{A-10}$$

where $\sin(\beta_n P - \theta) = 0$ for all values of n

$\cos(\beta_n P - \theta) = 1$ for all even values of n

$\cos(\beta_n P - \theta) = -1$ for all odd values of n

Therefore,

$$A_n = \frac{E_o l}{L}\,\frac{\sin\frac{\beta_n l}{2}}{\frac{\beta_n l}{2}} \tag{A-11}$$

$n = \pm$ odd integers

Or rewriting equation (A-11) in terms of the line parameters,

$$A_n = \frac{V_o}{L}\,\frac{\sin\left[\frac{\pi v_c l}{\lambda_o v_p}\left(1 + \frac{H}{P}\right) + \frac{\pi n l}{2P}\right]}{\left[\frac{\pi v_c l}{\lambda_o v_p}\left(1 + \frac{H}{P}\right) + \frac{\pi n l}{2P}\right]} \tag{A-12}$$

$n = \pm$ odd integers

Thus it is seen that the envelope of the magnitudes of the space

harmonic component is a function of the type $\frac{\sin x}{x}$. Only odd space harmonics exist for the folded strip-line structure. In solving for the magnitudes of the space harmonic components, the propagation constant, β_n, is defined in terms of the folded strip-line structural parameters.

APPENDIX B

Optimized Structural Parameters Utilized in Design of a Folded Strip-Line Oscillator

B-1 Optimizing l in Terms of Gain Parameter, C

The relationship which describes the interaction between an electron beam and the principal space harmonic component is given by J. R. Pierce[2] as,

$$C_n^3 = \frac{1}{4}\,\frac{E_n^2}{2\beta_n^2 P}\,\frac{I_B}{V_B} \qquad \text{(B-1)}$$

where C = gain parameter depending upon circuit and beam impedance

$\left(\frac{E_n^2}{2\beta_n^2 P}\right)$ = circuit impedance (parameter)

$\left(\frac{I_B}{V_B}\right)$ = beam impedance

E_n = field magnitude

P = power flow in circuit

Then assuming that only the principal space harmonic component is important, and substituting for the magnitude of the principal component from equation (A-11)[6],

$$C_n^3 = \frac{V_n^2 \sin^2 \frac{\beta_n l}{2}}{8 L^2 \frac{\beta_n^2 l^2}{4} \beta_n^2 P_n}\,\frac{I_B}{V_B} \qquad \text{(B-2)}$$

but

$$V_n = E_n l \qquad \text{(B-3)}$$

and

$$V_n^2 = 2 Z_0 = 240\pi \frac{l}{w} \qquad \text{(B-4)}$$

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Optimized Structural Parameters Utilized in Design of a Folded Strip-Line Oscillator

B-1 Optimizing K in Terms of Gain Parameter, C

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$$C^3 = \frac{1}{4}\frac{E^2}{\beta^2 P}\frac{I_0}{V_0} \qquad \text{(B-1)}$$

where C : gain parameter depending upon circuit and beam impedance

$\frac{E^2}{\beta^2 P}$: circuit impedance (parameter)

$\frac{V_0}{I_0}$: beam impedance

E : field magnitude

P : power flow in circuit

Then assuming that only the principal space harmonic component is important, and substituting for the magnitude of the principal component from equation (A-11)[5],

[illegible] (B-2)

but

[illegible] (B-3)

and

[illegible] (B-4)

then

$$C_u^{\,3} = \frac{120\pi \sin^2 \frac{\beta_u l}{2}}{\beta_u^{\,4} L^2 W l} \quad \frac{I_B}{V_B} \qquad \text{(B-5)}$$

or

$$C_u^{\,3} = K \; \frac{\sin^2 \frac{\beta_u l}{2}}{l} \qquad \text{(B-6)}$$

$$\text{where } K = \frac{120\,\pi}{L^2 \beta_u^{\,4} W} \quad \frac{I_B}{V_B} \qquad \text{(B-7)}$$

In order to maximize $C_u^{\,3}$ with respect to l, the partial with respect to l must be equated to 0, as follows,

$$\frac{\partial C_u^{\,3}}{\partial l} = K \left[\frac{\beta_u l \sin \frac{\beta_u l}{2} \cos \frac{\beta_u l}{2} - \sin^2 \frac{\beta_u l}{2}}{l^2} \right] = 0 \qquad \text{(B-8)}$$

But since

$$K \neq 0$$
$$l^2 \neq 0$$
$$\sin \frac{\beta_u l}{2} \neq 0$$

it follows that

$$\beta_u l = \tan \frac{\beta_u l}{2} \qquad \text{(B-9)}$$

The first solution of the above transcendental equation is[6]

$$\beta_u l = 133.6^\circ = 2.333 \text{ rad.} \qquad \text{(B-10)}$$

or

$$\frac{\beta_u l}{2} = 66.8^\circ = 1.166 \text{ rad.} \qquad \text{(B-11)}$$

Therefore,

$$M = \frac{\sin \frac{\beta_u l}{2}}{\frac{\beta_u l}{2}} = .79 \qquad \text{(B-12)}$$

Then, if the value of β_u in terms of the structural parameters is

substituted into equation (B-10), ℓ will be given by

$$\ell = \frac{2.333}{\frac{2\pi v_c}{\lambda_o v_p}\left(1+\frac{H}{P}\right)+\frac{\pi n}{P}} \tag{B-13}$$

where n = ± odd integers

Now, n must equal -1 in order that the principal backward space harmonic may fall within the largest portion of the envelope of the $\frac{\sin x}{x}$ function (See Fig. 4-1). Then assuming, as in Appendix A, that v_c and v_p are equivalent to those occurring in a parallel strip line with unity dielectric constant, it follows that,

$$\ell = \frac{2.333}{\frac{2\pi}{\lambda_o}\left[\frac{P+H+\frac{\lambda_o n}{2}}{P}\right]} \tag{B-14}$$

where n = ± odd integers

$$\ell = \frac{.743\ P}{\frac{2}{\lambda_o}\left\{(P+H)+n\right\}} \tag{B-15}$$

where n = ± odd integers

Then

$$\ell = \frac{.743\ P}{\frac{2}{\lambda_o}(P+H)-1} \tag{B-16}$$

if n = -1

B-2 <u>Beam Voltage in Terms of Space Harmonic Propagation Constant</u>

Since, for interaction, the electron velocity may be considered equal to the phase velocity, the propagation constant β_n may be expressed[6],

$$\beta_n = \frac{\omega}{v_{p_n}} = \frac{2\pi\ v_c}{\lambda_o v_{p_n}} \tag{B-17}$$

substituted into equation (B-10), will be given by

(B-13)

where n = odd integers

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(B-14)

where n = odd integers

(B-15)

where n = odd integers

Then

(B-16)

if n = -1

B- Beam Voltage in Terms of Space Harmonic Propagation Constant

Since, for interaction, the electron velocity may be considered equal to the phase velocity, the propagation constant may be expressed[6],

(B-17)

But

$$v_{p_n} = u_o = 5.93 \times 10^7 \sqrt{V_B} \qquad \text{(B-18)}$$

where u_o = electron velocity

Therefore

$$\beta_n = \frac{2\pi \times 506}{\lambda_o \sqrt{V_B}} \qquad \text{(B-19)}$$

B-3 <u>Gain Parameter in Terms of Folded Strip-Line Structure Parameters</u>

In equation (B-1) the gain parameter, C, was expressed in terms of structural parameters. Then from equation (A-11) the magnitude of the n^{th} space harmonic may be expressed,

$$E_n^2 = \frac{V_n^2}{L^2}\left[\frac{\sin\frac{\beta_n l}{2}}{\frac{\beta_n l}{2}}\right]^2 = \frac{V_n^2 M^2}{L^2} \qquad \text{(B-20)}$$

Then utilizing the value of β_n as expressed in equation (B-19), it follows that,

$$\frac{V_n^2}{2P_n} = 2Z_o = 240\pi \frac{l}{w} \qquad \text{(B-21)}$$

Then,

$$\begin{aligned} C_n^3 &= \frac{1}{4} \times \frac{240\pi l}{L^2 w} \times \frac{M^2 \lambda_o^2 V_B}{(2\pi \times 506)^2} \times \frac{I_B}{V_O} \\ &= 18.7 \times 10^{-6} \frac{M^2 \lambda_o^2}{L^2} \frac{I_B l}{w} \end{aligned} \qquad \text{(B-22)}$$

and[5]

$$C_n = .0266 \left(\frac{M \lambda_o}{L}\right)^{\frac{2}{3}} \left(\frac{I_B l}{w}\right)^{\frac{1}{3}} \qquad \text{(B-23)}$$

APPENDIX C

Quantitative Analysis of Backward Wave Interaction

C-1 Introduction

Quantitative analysis of backward wave interaction can be accomplished for small signal operation[3, 6]. Small signal operation is defined as:

$$i = i_0 + \tilde{i}(z)e^{j\omega t} \qquad \text{where } |\tilde{i}(z)| << |i_0| \tag{C-1}$$

$$\rho = \rho_0 + \tilde{\rho}(z)e^{j\omega t} \qquad \text{where } |\tilde{\rho}(z)| << |\rho_0| \tag{C-2}$$

$$u = u_0 + \tilde{u}(z)e^{j\omega t} \qquad \text{where } |\tilde{u}(z)| << |u_0| \tag{C-3}$$

The circuit-beam interaction of the dual phenomena which occurs is mathematically described, and later combined into a single relationship. The dual phenomena may be separately regarded as:

1. The Electronic Equation — The current induced in the electron beam is related to the field on the structure.
2. The Circuit Equation — The field induced on the structure is related to the current in the beam.

C-2 Development of the Electronic Equation

The force equation, $f = ma$, may here be applied to describe the effect of the structure field on the electron beam.

$$\frac{du}{dt} = -\frac{eE_t}{m} = -\frac{e}{m}E_t \tag{C-4}$$

Since $u = f(z,t)$, the total derivative will be,

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[illegible] (C-4)

Since $u = f(z,t)$, the total derivative will be,

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \tag{C-5}$$

but

$$\frac{dz}{dt} \approx u_0 \tag{C-6}$$

Then, where the frequency exponential is understood,

$$\frac{du}{dt} = j\omega \tilde{u}(z) + u_0 \frac{\partial \tilde{u}(z)}{\partial z} \tag{C-7}$$

Maxwell's continuity equation states that

$$\operatorname{div} \bar{i} = -\frac{\partial \rho}{\partial t} \tag{C-8}$$

since z is the only space variable considered, this may be written:

$$\frac{\partial i}{\partial z} = \frac{\partial \tilde{i}(z)}{\partial z} = -\frac{\partial \rho}{\partial t} = -j\omega \tilde{\rho}(z) \tag{C-9}$$

Then

$$\tilde{\rho}(z) = -\frac{1}{j\omega}\frac{\partial \tilde{i}(z)}{\partial z} \tag{C-10}$$

and

$$\frac{\partial \tilde{\rho}(z)}{\partial z} = -\frac{1}{j\omega}\frac{\partial^2 \tilde{i}(z)}{\partial z^2} \tag{C-11}$$

By definition

$$\bar{i} = \rho \bar{u} = (\rho_0 + \tilde{\rho}(z))(u_0 + \tilde{u}(z)) \tag{C-12}$$

and expanding,

$$i_0 + \tilde{i}(z) = \rho_0 u_0 + \rho_0 \tilde{u}(z) + u_0 \tilde{\rho}(z) + \tilde{\rho}(z)\tilde{u}(z) \tag{C-13}$$

Then, since for small signal theory the second order term may be

disregarded[6],

$$\tilde{i}(z) = \rho_0 \tilde{u}(z) + u_0 \tilde{\rho}(z) \tag{C-14}$$

Now solving for $\tilde{u}(z)$,

$$\tilde{u}(z) = \frac{\tilde{i}(z) - u_0 \tilde{\rho}(z)}{\rho_0} = \frac{\tilde{i}(z)}{\rho_0} + \frac{u_0}{j\omega\rho_0} \frac{\partial \tilde{i}(z)}{\partial z} \tag{C-15}$$

and taking the partial derivative with respect to z

$$\frac{\partial \tilde{u}(z)}{\partial z} = \frac{1}{\rho_0} \frac{\partial \tilde{i}(z)}{\partial z} - \frac{u_0}{\rho_0} \frac{\partial \tilde{\rho}(z)}{\partial z} \tag{C-16}$$

$$= \frac{1}{\rho_0} \frac{\partial \tilde{i}(z)}{\partial z} + \frac{u_0}{j\omega\rho_0} \frac{\partial^2 \tilde{i}(z)}{\partial z^2} \tag{C-17}$$

Then substituting in the force equation[6]

$$j\frac{\omega}{\rho_0}\tilde{i}(z) + \frac{u_0}{\rho_0}\frac{\partial \tilde{i}(z)}{\partial z} + \frac{u_0}{\rho_0}\frac{\partial \tilde{i}(z)}{\partial z} + \frac{u_0^2}{j\omega\rho_0}\frac{\partial^2 \tilde{i}(z)}{\partial z^2} = \frac{e}{m}E_t \tag{C-18}$$

Multiplying through by $\frac{j\omega\rho_0}{u_0^2}$, it follows that,

$$\frac{\partial^2 \tilde{i}(z)}{\partial z^2} + 2j\frac{\omega}{u_0}\frac{\partial \tilde{i}(z)}{\partial z} - \frac{\omega^2}{u_0^2}\tilde{i}(z) = \frac{j\omega\rho_0 e}{u_0^2 m}E_t \tag{C-19}$$

Now, the average electron beam propagation constant may be defined in terms of the average beam velocity, and will be,

$$\beta_e = \frac{\omega}{u_0} \tag{C-20}$$

Also, if the total field $E_t(z)$ is assumed to be composed of the exciting field on the circuit, $E_c(z)$, plus the field due to all the electrons in the space charge adjacent to point being considered, then

$$E_T(z) = E_c(z) + E_{sc}(z) \tag{C-21}$$

disregarded[6],

(6-14)

Now solving for

(6-15)

and taking the partial derivative with respect to z

(6-16)

(6-17)

Then substituting in the force equation[6]

(6-18)

Multiplying through by , it follows that,

(6-19)

Now, the average electron beam propagation constant may be defined in terms of the average beam velocity, and will be,

(6-20)

Also, if the total field is assumed to be composed of the exciting field on the circuit, , plus the field due to all the electrons in the space charge adjacent to point being considered, then

(6-21)

But, again from Maxwell's equations, the space charge field may be defined

$$\overline{D} = \epsilon \, \overline{E}_{sc} \tag{C-22}$$

and

$$\operatorname{div} \overline{D} = \rho \tag{C-23}$$

or

$$\operatorname{div} \overline{E}_{sc} = \frac{\rho}{\epsilon} \tag{C-24}$$

Again, since the only space variable is z,

$$\frac{\partial E_{sc}(z)}{\partial z} = \frac{\rho}{\epsilon} \tag{C-25}$$

then

$$j\omega \frac{\partial E_{sc} z}{\partial z} = \frac{1}{\epsilon} \frac{\partial \tilde{\rho}(z)}{\partial t} \tag{C-26}$$

and substituting from the continuity equation

$$\frac{\partial E_{sc}(z)}{\partial z} = -\frac{1}{j\omega\epsilon} \frac{\partial \tilde{i}(z)}{\partial z} \tag{C-27}$$

Now if the boundary conditions for $z=0$, the plane where the beam enters the structure, are studied it will be seen that there are no a.c. components of current or beam velocity and therefore,

$$\tilde{i}(z) = 0 \quad \text{at } z = 0 \tag{C-28}$$

$$\tilde{u}(z) = 0 \quad \text{at } z = 0 \tag{C-29}$$

and

$$\tilde{\rho}(z) = 0 \quad \text{at } z = 0 \tag{C-30}$$

since, in the entire beam,

$$\tilde{i}(z) = u_0 \tilde{\rho}(z) + \rho_0 \tilde{u}(z) \tag{C-13}$$

Now, if both $\tilde{u}(z) = 0$ and $\tilde{\rho}(z) = 0$ at $z = 0$ then the space charge field $E_{sc}(z)$

will be zero at z_{∞} and the constant of integration will be zero thus:

$$\int_0^z \frac{\partial E_{sc}(z)}{\partial z} dz = -\frac{1}{j\omega\epsilon}\int_0^z \frac{\partial \tilde{i}(z)}{\partial z} dz \tag{C-31}$$

and

$$E_{sc}(z) = -\frac{\tilde{i}(z)}{j\omega\epsilon} \tag{C-32}$$

Also, if a quantity h, termed the plasma wave number, is defined as

$$h^2 = \frac{e\rho_0}{m\epsilon u_0^2} \tag{C-33}$$

the original force equation may be simplified as follows,

$$\frac{\partial^2 \tilde{i}(z)}{\partial z} + 2j\beta_e \frac{\partial \tilde{i}(z)}{\partial z} - \beta_e^2 \tilde{i}(z) = \frac{j\omega\rho_0 e}{u_0^2 m} E_c + \left(\frac{j\omega\rho_0 e}{u_0^2 m}\right)\left(-\frac{1}{j\omega\epsilon}\right)\tilde{i}(z) \tag{C-34}$$

or,

$$\frac{\partial^2 \tilde{i}(z)}{\partial z} + 2j\beta_e \frac{\partial \tilde{i}(z)}{\partial z} - \left(\beta_e^2 - h^2\right)\tilde{i}(z) = \frac{j\omega\rho_0 e}{u_0^2 m} E_c(z) \tag{C-35}$$

This is the Electronic Equation.

C-3 Development of the Circuit Equation

In describing the field induced on the circuit due to the current in the beam, the following assumptions are made:

1. The only fields interacting with the beam are those associated with the space harmonic component which has its phase velocity close to the electron beam velocity.
2. A single mode is propagating on the circuit and has opposite group and phase velocities.
3. For propagation in the positive z direction, $\gamma = \alpha - j\beta$ such that

will be zero at z = 0 and the constant of integration will be zero also:

[illegible] (C-31)

and

[illegible] (C-32)

Also, if a quantity h, termed the plasma wave number, is defined as

[illegible] (C-33)

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1. The only fields interacting with the beam are those associated with the space harmonic component which has its phase velocity close to the electron beam velocity.
2. A single mode is propagating on the circuit and has opposite group and phase velocities.
3. For propagation in the positive z direction, [illegible] such that

β will be a positive number.

4. The line is perfectly terminated at each end.

5. A source exists at $z = L$ end, as shown, such that a longitudinal field E_o exists at $z = L$.

If an infinitesimal source generator, $A(z)dz$, an infinite number of which lie along the beam, is defined in terms of the current at a point, such that

$$A(z) = G\, i(z) \tag{C-36}$$

then, it has been shown by Bernier[4] and Pierce[2] that

$$G = -\beta^2 Z = -\frac{\beta^2 E^2}{2\beta^2 P} \tag{C-37}$$

because

$$Z = \frac{E^2}{2\beta^2 P} \tag{C-38}$$

Then,

$$A(z) = -\beta^2 Z\, i(z) \tag{C-39}$$

Now, the field at any point z will be the sum of the effects from the distributed elements plus any applied field[3]. Then with the factor understood, the total field may be stated thus:

$$E(z) = E_o e^{-\gamma z} + \frac{1}{2}\int_0^z A(x) e^{-\gamma(z-x)} dx + \frac{1}{2}\int_z^L A(x) e^{\gamma(z-x)} dx \tag{C-40}$$

Differentiating with respect to z results in

$$\frac{dE(z)}{dz} = -\gamma\left[E_o e^{-\gamma z} + \frac{1}{2}\int_0^z A(x) e^{-\gamma(z-x)} dx - \frac{1}{2}\int_z^L A(x) e^{\gamma(z-x)} dx\right] + \frac{A(z)}{2} - \frac{A(z)}{2} \tag{C-41}$$

Differentiating again with respect to z results in

$$\frac{d^2E(z)}{dz^2} = \gamma^2\left[E_0e^{-\gamma z} + \frac{1}{2}\int_0^z A(x)e^{-\gamma(z-x)}dx + \frac{1}{2}\int_z^L A(x)e^{\gamma(z-x)}dx\right] - \gamma\frac{A(z)}{2} - \gamma\frac{A(z)}{2} \quad \text{(C-42)}$$

Noting that the term in the brackets is the original $E(z)$

$$\frac{d^2E(z)}{dz^2} - \gamma^2E(z) = -\gamma A(z) \quad \text{(C-43)}$$

or

$$\frac{d^2E(z)}{dz^2} - \gamma^2E(z) = \gamma\beta^2 Z\, i(z) \quad \text{(C-44)}$$

This, of course, is equivalent to the integral equation (C-40) above, and it is, therefore, the Circuit Equation[3, 6].

C-4 Solution of the Combined Equation

The field on the structure, $E(z)$, in terms of beam-circuit parameters, may now be obtained by simultaneous solution of the Electronic Equation, (C-35), and the Circuit Equation, (C-44). Since each equation is of the second order, four solutions are to be expected. The LaPlace Transform will be used because of its inherent simplicity.[3] Then if

$$I(s) = \int_0^\infty i(z)e^{-sz}dz \quad \text{(C-45)}$$

$$E(s) = \int_0^\infty E(z)e^{-sz}dz \quad \text{(C-46)}$$

and the boundary conditions are

$$i(z) = 0 \quad \text{at } z = 0 \quad \text{(C-27)}$$

$$\frac{\partial i(z)}{\partial z} = 0 \quad \text{at } z = 0 \quad \text{(C-47)}$$

$$E(0) = \text{value of circuit field at } z = 0 \quad \text{(C-48)}$$

*

$$E'(0) = \text{value of derivative of circuit field at } z = 0 \tag{C-49}$$

The transforms of the simultaneous equations are then

$$\left[s^2 + 2j\beta_e s - (\beta_e^2 - h^2)\right] I(s) = \frac{j\omega e \rho_0}{m u_0^2} E(s) \tag{C-50}$$

and

$$(s^2 + \beta^2) E(s) - (s - j\beta) E(0) = -j\beta^3 Z\, I(s) \tag{C-51}$$

where $\gamma = -j\beta$

Then solving the transforms for $I(s)$

$$I(s) = \frac{K E(s)}{j\beta^3 Z\left[(s + j\beta_e)^2 + h^2\right]} \tag{C-52}$$

$$I(s) = \frac{(s^2 + \beta^2) E(s) - (s - j\beta) E(0)}{-j\beta^3 Z} \tag{C-53}$$

where $K = \dfrac{I_B}{2V_B} \dfrac{E^2}{2\beta^2 P} \beta^3 \beta_e$ (C-54)

$= \dfrac{Z}{2Z_0} \beta^3 \beta_e$ (C-55)

where $Z_0 = \dfrac{V_B}{I_B}$ (Beam Impedance)

$Z = \dfrac{E^2}{2\beta^2 P}$ (Circuit Impedance)

or K may be written

$$K = -\omega_E \beta^3 Z h^2 \tag{C-56}$$

Eliminating I(s) from equations (C-52) and (C-53),

$$E(s) = \frac{\left[(s + j\beta_e)^2 + h^2\right]\left[s - j\beta\right] E(0)}{\left[(s + j\beta_e)^2 + h^2\right]\left[s^2 + \beta^2\right] + K} \tag{C-57}$$

The inverse transform may now be taken by the partial fraction expansion technique[8]. Hence,

$$E(z) = \mathcal{L}^{-1} E(s) = \mathcal{L}^{-1} \frac{A(s)}{B(s)} = \sum_{\substack{k=1 \\ 0 \leq z}}^{8} \frac{A(s_k)}{B'(s_k)} e^{s_k z} \tag{C-58}$$

where $s_1, s_2, \ldots s_8$ are the roots of $B(s)$

and $\left.\frac{dB(s)}{ds}\right|_{s=s_k} = B'(s_k)$

Since B(s) is of the fourth order, there will be four roots, and hence four exponentials in the inverse transform of equation (C-57)[3, 6]. Therefore,

$$E(z) = E(0)\left[\frac{A(\Gamma_1)}{B'(\Gamma_1)} e^{\Gamma_1 z} + \frac{A(\Gamma_2)}{B'(\Gamma_2)} e^{\Gamma_2 z} + \frac{A(\Gamma_3)}{B'(\Gamma_3)} e^{\Gamma_3 z} + \frac{A(\Gamma_4)}{B'(\Gamma_4)} e^{\Gamma_4 z}\right] \tag{C-59}$$

where Γ_n = roots of B(s) n = 1, 2, 3, 4

and

$$B(s) = \left[(s + j\beta_e)^2 + h^2\right]\left[s^2 + \beta^2\right] + K \tag{C-60}$$

and

$$B'(\Gamma_1) = \left[(\Gamma_1 - \Gamma_2)(\Gamma_1 - \Gamma_3)(\Gamma_1 - \Gamma_4)\right] \tag{C-61}$$

and

$$A(\Gamma_1) = \left[(\Gamma_1 + j\beta_e)^2 + h^2\right]\left[\Gamma_1 - j\beta\right] \tag{C-62}$$

If a new magnitude term is defined, that is,

$$C(\Gamma_n) = \frac{A(\Gamma_n)}{B'(\Gamma_n)(\Gamma_n - j\beta)} \tag{C-63}$$

Equation (C-59) may be rewritten as

$$E(z) = E(0)\left[\sum_{n=1}^{4} C(\Gamma_n)(\Gamma_n - j\beta) e^{\Gamma_n z}\right] \tag{C-64}$$

The inverse transform may now be taken by the partial fraction expansion technique[8]. Hence,

[illegible] (C-58)

where [illegible] are the roots of [illegible]

and [illegible]

Since $S(s)$ is of the fourth order, there will be four roots, and hence four exponentials in the inverse transform of equation (C-57)[3, 6].

Therefore,

[illegible] (C-59)

where [illegible] roots of $S(s)$ $n = 1, 2, 3, 4$

and

[illegible] (C-60)

and

[illegible] (C-61)

and

[illegible] (C-62)

If a new magnitude term is defined, that is,

[illegible] (C-63)

Equation (C-59) may be rewritten as

[illegible] (C-64)

Before obtaining the roots of equation (C-60), it will be better to evaluate the constant $E(o)$. The process will involve stating $\tilde{\tau}(z)$ in terms of $E(z)$ using equations (C-43) and (C-64), and then using $\tilde{\tau}(z)$ so stated, in the integral equation (C-39). Rewriting equation (C-43) and substituting $E(z)$ from equation (C-64), $\tilde{\tau}(z)$ becomes

$$\tilde{\tau}(z) = \frac{1}{r\beta^3 Z} \frac{\partial^2 E(z)}{\partial z^2} - \frac{r}{\beta^2 Z} E(z) \qquad \text{(C-65)}$$

$$= \frac{jE(o)}{\beta Z} \sum_{n=1}^{4} \left[\frac{\Gamma_n^2}{\beta^2}+1\right]\left[C(\Gamma_n)(\Gamma_n - j\beta)e^{\Gamma_n z}\right]$$

But

$$A(x) = -\beta^2 Z \tilde{\tau}(x) \qquad \text{(C-39)}$$

where the variable x replaces z

Then

$$A(x) = \frac{-jE(o)}{\beta} \sum_{n=1}^{4} \left[\Gamma_n^2 + \beta^2\right]\left[C(\Gamma_n)(\Gamma_n - j\beta)e^{\Gamma_n x}\right] \qquad \text{(C-66)}$$

Now, evaluating equation (C-40) at z = o

$$E(o) = E_o - j\frac{E(o)}{2\beta} \sum_{n=1}^{4} \left[\Gamma_n^2 + \beta^2\right]\left[C(\Gamma_n)(\Gamma_n - j\beta)\right] \int_0^L e^{(\Gamma_n + j\beta)x} dx \qquad \text{(C-67)}$$

or

$$E(o) = E_o - \frac{KE(o)}{j\,2\beta} \sum_{n=1}^{4} \frac{(\Gamma_n - j\beta)(1 - e^{(\Gamma_n + j\beta)L})}{(\Gamma_n + j\beta)\, B'(\Gamma_n)} \qquad \text{(C-68)}$$

or

$$E(0) = \frac{E_0}{1 + \frac{K}{j2\beta} \sum_{n=1}^{4} \frac{(\Gamma_n - j\beta)\left[1 - e^{(\Gamma_n + j\beta)L}\right]}{(\Gamma_n + j\beta)\, B'(\Gamma_n)}} \tag{C-69}$$

Once the roots Γ_n are known, $E(0)$ may be determined exactly in terms of input field and beam circuit parameters.

With the initial condition $E(0)$ described, it only remains to solve equation (C-60). From the roots, the field on the structure may be written[3, 6]. In order to solve for the roots, the right member of the equation (C-60) must be equated to zero, thus,

$$\left[(s + j\beta_e)^2 + \Omega^2\right]\left[s^2 + \beta^2\right] + K^4 = 0 \tag{C-60}$$

Time restriction prevents the continued detailed analysis which is the purpose of this paper. But to continue in brief: to obtain the exact roots of equation (C-60) it would be best to supply numerical values and solve for the roots using one of the well known methods. However, it has been shown, using suitable substitutions and variable changes, that one of the roots will be[3, 6]

$$\Gamma_4 \approx j\beta \tag{C-70}$$

This is based on the demonstrable assumption that K is quite small, that is

$$K \ll 1 \tag{C-71}$$

Further, if the three other roots are defined,

$$\Gamma_n = -j\beta + j\frac{\eta_n}{L} \tag{C-72}$$

where n = 1, 2, 3

η = incremental propagation constant

then equation (C-60) may be rewritten[3]

$$\eta^3 + 2\theta\eta^2 + (\theta^2 - H^2)\eta + \frac{K}{2\beta} = 0 \qquad \text{(C-73)}$$

where $\theta = (\beta_e - \beta)L$

$H^2 = h^2 L^2$

Now if equation (C-64) is written in terms of the new incremental propagation constant, η_m , and the value for $E_{(o)}$ as determined in equation (C-69) is also substituted, the total field, $E_{(z)}$, will become[3, 6]

$$E_{(z)} = \frac{\dfrac{\eta_3 - \eta_2}{\eta_1} e^{j\eta_1 \frac{z}{L}} + \dfrac{\eta_1 - \eta_3}{\eta_2} e^{j\eta_2 \frac{z}{L}} + \dfrac{\eta_2 - \eta_1}{\eta_3} e^{j\eta_3 \frac{z}{L}}}{\dfrac{\eta_3 - \eta_2}{\eta_1} e^{j\eta_1} + \dfrac{\eta_1 - \eta_3}{\eta_2} e^{j\eta_2} + \dfrac{\eta_2 - \eta_1}{\eta_3} e^{j\eta_3}} E_0 \qquad \text{(C-74)}$$

Note that where $z = L$, $E_{(z)}$ is exactly equal to the applied field E_0 .

The start oscillation condition occurs when the denominator of equation (C-74) is set equal to zero[3], thus

$$\frac{\eta_3 - \eta_2}{\eta_1} e^{j\eta_1} + \frac{\eta_1 - \eta_3}{\eta_2} e^{j\eta_2} + \frac{\eta_2 - \eta_1}{\eta_3} e^{j\eta_3} = 0 \qquad \text{(C-75)}$$

If no loss is present, one root of equation (C-73) will be real; let this root be η_1 . Then the remaining roots may be expressed in terms of η_1 , thus[3],

$$\eta_2 = -\theta - \frac{\eta_1}{2} + j\frac{1}{2}\sqrt{4\theta\eta_1 + 3\eta_1^2 - 4H^2}$$
$$\eta_3 = -\theta - \frac{\eta_1}{2} - j\frac{1}{2}\sqrt{4\theta\eta_1 + 3\eta_1^2 - 4H^2} \qquad \text{(C-76)}$$

Then by substituting into equation (C-75) and equating the real and imaginary parts to zero, the start oscillation conditions may be stated

$$\cos\left(\theta + \frac{3\eta_1}{2}\right) + \frac{2\eta_1(\theta + \eta_1)}{(\theta + \eta_1)^2 - H^2}\cosh\sqrt{\theta\eta_1 + \frac{3\eta_1^2}{4} - H^2} = 0 \qquad \text{(C-77)}$$

$$\gamma^3 + 2\Theta\gamma^2 + (\Theta^2 - \mu^2)\gamma + [illegible] = 0 \qquad \text{(C-73)}$$

where $\Theta = [illegible]$

$\mu^2 = [illegible]$

Now if equation (C-54) is written in terms of the new incremental propagation constant, [illegible] , and the value for [illegible] as determined in equation (C-59) is also substituted, the total field, [illegible] , will become[3, 6]

$$[illegible] \qquad \text{(C-74)}$$

Note that where [illegible] , [illegible] is exactly equal to the applied field [illegible] .

The start oscillation condition occurs when the denominator of equation (C-74) is set equal to zero[3], thus

$$[illegible] \qquad \text{(C-75)}$$

If no loss is present, one root of equation (C-75) will be real; let this root be [illegible] . Then the remaining roots may be expressed in terms of [illegible] , thus[3],

$$[illegible] \qquad \text{(C-76)}$$

Then by substituting into equation (C-75) and equating the real and imaginary parts to zero, the start oscillation conditions may be stated

$$[illegible] \qquad \text{(C-77)}$$

$$\sin\left(\theta + \frac{3q_1}{2}\right) + \frac{\theta(\theta + q_1) + H^2}{(\theta + q_1)^2 - H^2}\left[\frac{q_1}{\sqrt{\theta q_1 + \frac{3q_1^2}{4} - H^2}}\right]\sinh\sqrt{\theta q_1 + \frac{3q_1^2}{4} - H^2} = 0 \qquad \text{(C-78)}$$